Rigidity theorems of generalized Veronese surfaces in Sn and Pn(C)

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博士論文

Rigidity theorems of generalized Veronese surfaces in S^n and $P^n(C)$

（球面と複素射影空間内の一般化されたヴェロネーズ曲面の剛性定理について）

尾方隆司

平成6年

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Takashi Ogata

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Contents

Introduction

Rigidity problems arise naturally whenever a Riemannian manifold M is immersed into another Riemannian manifold $\bar{M}$. To be more specific, two given isometric immersions of M into $\bar{M}$ are said to be *congruent* if they differ only by an isometry of the ambient space $\bar{M}$. Then the *rigidity* is a problem concerning the uniqueness of the isometric immersions in question up to their congruence.

The main objective of this thesis is the rigidity problem of generalized Veronese surfaces in $\mathbf{S}^n$ and $\mathbf{P}^n(\mathbf{C})$, where $\mathbf{S}^n$ denotes the n-dimensional unit sphere of the (n+1)-dimensional Euclidean space $\mathbf{R}^{n+1}$ and $\mathbf{P}^n(\mathbf{C})$ the n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4ρ, respectively. We shall prove, from the point of view of pinching theorems concerning the Gaussian curvature, several characterizing theorems for generalized Veronese surfaces in $\mathbf{S}^n$ or in $\mathbf{P}^n(\mathbf{C})$, which are closely related to their rigidity properties.

More precisely, let $\mathbf{S}^2(\mathrm{K})$ be a 2-sphere of constant Gaussian curvature K. In [10], Borůvka constructed, by making use of spherical harmonic polynomials of degree k, a family of isometric minimal immersions

$$\varphi_k : \mathbf{S}^2(\mathrm{K}(k)) \longrightarrow \mathbf{S}^{2k},$$

where we denote for simplicity

$$\mathrm{K}(k) = \frac{2}{k(k+1)}$$

for a positive integer k. For example, when $k = 2$, the immersion

$$\varphi_2 : \mathbf{S}^2(\frac{1}{3}) \longrightarrow \mathbf{S}^4$$

is defined by

$$\varphi_2(x, y, z) = \frac{1}{\sqrt{3}}(yz, zx, xy, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2)).$$

This immersion φ_2 is called *the Veronese immersion*, which has been the most familiar isometric minimal immersion into $\mathbf{S}^4$. The other immersions φ_k $(k \geq 3)$ are defined as generalizatins of φ_2 and are called the *generalized Veronese immersions* in $\mathbf{S}^{2k}$.

Later it was proved by Calabi [11] that any full isometric minimal immersion of $\mathbf{S}^2(\mathrm{K})$ into $\mathbf{S}^n$ is congruent to some φ_k and hence there exists a positive integer k such that $\mathrm{K} = \mathrm{K}(k)$ and $n = 2k$.

Then the following conjecture has been proposed by Simon in 1980 to characterize these immersions φ_k more quantitatively:

Conjecture (Simon [38]). *Let $x : \mathbf{M} \longrightarrow \mathbf{S}^n$ be a full isometric minimal immersion of a closed connected oriented 2-dimensional Riemannian manifold $\mathbf{M}$ into $\mathbf{S}^n$. If the Gaussian curvature K of $\mathbf{M}$ satisfies*

$$\mathrm{K}(k+1) \leq \mathrm{K} \leq \mathrm{K}(k)$$

for some positive integer k, then either $\mathrm{K} = \mathrm{K}(k)$ or $\mathrm{K} = \mathrm{K}(k+1)$ and x is congruent to φ_k or φ_{k+1}, respectively.

This conjecture has been proved affirmatively in each of the following cases:

(1) (Calabi [11], 1967) If $x : \mathbf{M} \longrightarrow \mathbf{S}^{2m}$ is a full isometric minimal immersion and $\mathrm{K} \geq \mathrm{K}(\mathrm{m})$, then $\mathrm{K} = \mathrm{K}(\mathrm{m})$ and x is congruent to φ_m.

(2) (Benko et al. [4], 1979) If $x : \mathbf{M} \longrightarrow \mathbf{S}^n$ is a full isometric minimal immersion with $1 \geq \mathrm{K} \geq \mathrm{K}(2)$, then $\mathrm{K} = 1$ or $\mathrm{K} = \mathrm{K}(2)$.

(3) (Ogata [32], 1985) If $x : \mathbf{M} \longrightarrow \mathbf{S}^n$ is a full isometric minimal immersion with $\mathrm{K}(2) \geq \mathrm{K} \geq \mathrm{K}(3)$, then $\mathrm{K} = \mathrm{K}(2)$ or $\mathrm{K} = \mathrm{K}(3)$.

(4) (Ogata [33], 1987, Bolton et al. [5], 1988, Itoh [21], 1988) If each point of $\mathbf{M}$ is a regular point of higher order, then the Conjecture is true.

(5) (Bolton and Woodward [6], 1988) If a full isometric minimal immersion $x : \mathrm{M} \longrightarrow \mathrm{S}^n$ has S^1-symmetry, then the Conjecture is true.

Note that (4) generalizes (2) and (3). The condition (5) implies (4). Hence, (5) is reduced to (4).

In Chapter I of this thesis we shall be mainly concerned with Simon's conjecture and prove the case (4) by using results due to Ogata [32 and 33]. In fact, we shall prove the following two curvature pinching theorems.

Theorem (Theorem 3.3). *Let $x : \mathrm{M} \longrightarrow \mathrm{S}^n$ be a full isometric minimal immersion of a complete connected oriented 2-dimensional Riemannian manifold M into S^n $(n \geq 3)$. If the Gaussian curvature K of M satisfies*

$$\mathrm{K}(k) \leq \mathrm{K} \leq 1,$$

then either (1) $x = \varphi_k$ *and* $\mathrm{K} = \mathrm{K}(k)$, *or* (2) $n < 2k$.

Theorem (Theorem 3.4). *Let $x : \mathrm{M} \longrightarrow \mathrm{S}^n$ be a full isometric minimal immersion of a complete connected oriented 2-dimensional Riemannian manifold M into S^n $(n \geq 3)$. If the Gaussian curvature K of M satisfies*

$$\delta \leq \mathrm{K} \leq \mathrm{K}(k)$$

for some positive number δ, and if each point of M is a regular point of order k, then either (1) $x = \varphi_k$ *and* $\mathrm{K} = \mathrm{K}(k)$, *or* (2) $n > 2k$.

It should be remarked that Theorem (Theorem 3.3) for $k = 3$ and Theorem (Theorem 3.4) for $k = 2$ have been proved by Ogata [32]. Combining Theorem (Theorem 3.3) with Theorem (Theorem 3.4) then proves the case (4).

In order to prove Theorems(Theorem 3.3 and Theorem 3.4), we first define for a given immersion its higher order fundamental forms and calculate the Laplacian of the square of their lengths for a minimal surface in $\mathbf{S}^n$. Since a bit difference appears between the formulas of the Laplacian for the second fundamental form and that for the third fundamental form, we shall calculate them separately. However, the Laplacian in the higher order cases can be computed inductively.

At the end of Chapter I we shall construct a 1-parameter family of minimal immersions of a 2-sphere $\mathbf{S}^2$ into $\mathbf{S}^4$, which gives rise to infinitely many examples of minimal surfaces in $\mathbf{S}^4$ with $1/6 \leq \mathrm{K} \leq 1$. This shows that the hypotheses on the Gaussian curvature K in Theorems (Theorem 3.3 and Theorem 3.4) are sharp in the case where $k = 1$ in the Conjecture.

In Chapter II we shall prove several rigidity theorems for minimal surfaces in $\mathbf{P}^n(\mathbf{C})$. To this end we shall introduce the notion of the Kaehler function, which is defined for general submanifolds in a Kaehler manifold in the following way.

Let $\mathbf{X}$ be a Kaehler manifold with complex structure J and Kaehler metric $<\,,\,>$. Let $x : \mathbf{M} \longrightarrow \mathbf{X}$ be an isometric immersion of a complete connected oriented 2-dimensional Riemannian manifold $\mathbf{M}$ into $\mathbf{X}$. By means of the differential $\mathrm{d}x$, locally we may identify a vector field v on $\mathbf{M}$ with $\mathrm{d}x(\mathrm{v})$ along the image of x. For an oriented local orthonormal frame field $\{\mathrm{e}_1, e_2\}$ in $\mathbf{M}$, the *Kaehler function* $cos(\alpha)$ of x is then defined by

$$cos(\alpha) = < Je_1, e_2 > .$$

The Kaehler function is an invariant of the immersion x, which measures the deviation of x from being a holomorphic map. Indeed, x is holomorphic if and only if $cos(\alpha) = 1$ on $\mathbf{M}$, while x is anti-holomorphic if and only if $cos(\alpha) = -1$ on $\mathbf{M}$.

It was Chern and Wolfson [18] who pointed out that the Kaehler function of x plays an important role in the study of minimal surfaces in $\mathbf{X}$. Based on their study we would like to examine all isometric minimal immersions of constant Kaehler function in $\mathbf{X}$.

In $\mathbf{P}^n(\mathbf{C})$ the following examples of minimal surfaces of constant Kaehler function have been given independently by Bando and Ohnita [2] and Bolton et al.[5]. Let k be an integer with $0 \leq k \leq$ n. Then there exists a full isometric minimal immersion

$$\varphi_{n,k} : \mathbf{S}^2(\mathrm{K}_{n,k}) \longrightarrow \mathbf{P}^n(\mathbf{C}),$$

where

$$\mathrm{K}_{n,k} = \frac{4\rho}{n + 2k(n-k)}.$$

Moreover, each $\varphi_{n,k}$ has holomorphic rigidity in the sense that for any isometric immersion φ of $\mathbf{S}^2(\mathrm{K}_{n,k})$ into $\mathbf{P}^n(\mathbf{C})$, there exists a holomorphic isometry g of $\mathbf{P}^n(\mathbf{C})$ such that $\varphi = g\varphi_{n,k}$. The Kaehler function $cos(\alpha_{n,k})$ of $\varphi_{n,k}$ is given by

$$cos(\alpha_{n,k}) = \frac{n - 2k}{n + 2k(n-k)},$$

for which the following formula holds:

$$\mathrm{K}_{n,k} = \frac{2(1 - (2k+1)cos(\alpha_{n,k}))\rho}{k(k+1)}.$$

These immersions $\varphi_{n,k}$ are also called the *generalized Veronese immersions* of $\mathbf{S}^2(\mathrm{K}_{n,k})$ into $\mathbf{P}^n(\mathbf{C})$ in this thesis.

In [31] Ohnita has characterized minimal surfaces in $\mathbf{P}^n(\mathbf{C})$ whose Kaehler function and Gaussian curvature are both constant. In fact he proved the following theorem:

Theorem (Ohnita [31]). *Let* $\mathbf{M}$ *be a complete connected oriented 2-dimensional Riemannian manifold and* $x : \mathbf{M} \longrightarrow \mathbf{P}^n(\mathbf{C})$ *a full isometric minimal immersion of* $\mathbf{M}$ *into* $\mathbf{P}^n(\mathbf{C})$. *Assume that the Gaussian curvature* K *of* $\mathbf{M}$ *and the Kaehler function* $cos(\alpha)$ *of* x *are both constant on* $\mathbf{M}$. *Then the following holds.*

(1) *If* $\mathrm{K} > 0$, *then there exists some* k *with* $0 \leq k \leq n$ *such that* $\mathrm{K} = \mathrm{K}_{n,k}$, $cos(\alpha) = cos(\alpha_{n,k})$ *and* $\varphi(\mathbf{M})$ *is an open submanifold of* $\varphi_{n,k}(\mathbf{S}^2(\mathrm{K}))$.

(2) *If* $K = 0$, *then* $cos(\alpha) = 0$, *that is,* φ *is totally real.* (*Such* φ*'s have already been classified by Kenmotsu* [24].)

(3) *The case of* $K < 0$ *is impossible.*

It should be remarked that Kenmotsu [25] and Ohnita [31] conjecture that the above theorem remains true without assuming the constancy of the Kaehler function.

On the other hand, Bolton et al. [5] conjecture that if the Kaehler function is constant, then the Gaussian curvature is also constant, provided the immersion is neither holomorphic, anti-holomorphic nor totally real. They prove the conjecture affirmatively for $n \leq 4$. We shall discuss their conjecture under some additional assumptions and prove the following rigidity theorems (Ogata [34]):

Theorem (Theorem 7.10). *Let* M *be a complete connected oriented 2-dimensional Riemannian manifold and* $x : M \longrightarrow \mathbf{P}^n(\mathbf{C})$ *a full isometric minimal immersion of* M *into* $\mathbf{P}^n(\mathbf{C})$ *with constant Kaehler function* $cos(\alpha)$, *which is neither holomorphic, anti-holomorphic nor totally real. If the J-invariant first osculating space of* x *is of constant dimension on* M *and the Gaussian curvature* K *of* M *is strictly positive and satisfies*

$$\frac{(1 - 7cos(\alpha))\rho}{6} \leq K$$

on M, *then* K *is constant on* M. *Moreover,* x *is congruent to either* $\varphi_{n,1}$, $\varphi_{n,2}$ *or* $\varphi_{n,3}$.

A point p of M is called a J-regular point of order m if there exists the $4m$-dimensional J-invariant m-th osculating space in a neighbourhood of p. We say that M is a J-regular manifold if each J-invariant higher order normal bundle is of constant rank on M (cf. Section 8). Generalizing Theorem (Theorem 7.10), we shall also prove the following (Ogata [35]).

Theorem (Theorem 10.4). *Let* $x : \mathrm{M} \longrightarrow \mathbf{P}^n(\mathbf{C})$ *be as in Theorem (Theorem 7.10) and* $s = [n/2 - 1] - 1$, ([a] *denotes the integer part of* a). *If* M *is a J-regular manifold and if* K *and* $cos(\alpha)$ *satisfy*

$$\frac{2\{1-(2s+3)cos(\alpha)\}\rho}{(s+1)(s+2)} \leq \mathrm{K},$$

then K *is constant on* M *so that* x *is congruent to either* $\varphi_{n,1}, \ldots, \varphi_{n,s}$ *or* $\varphi_{n,s+1}$.

In Chapter III we shall study an isometric non-minimal immersion $x : \mathrm{M} \longrightarrow \mathbf{P}^2(\mathbf{C})$ of a simply connected 2-dimensional Riemannian manifold M into the 2-dimensional complex projective space. If x is minimal, then several examples of such immersions are known and a variety of results characterizing them has been obtained (cf. [20],[22],[30],[34] and [35]). It is then natural to ask whether these results valid for minimal surfaces can be generalized to the case of surfaces having a weaker condition than the minimality.

When the ambient manifold **X** is a Riemannian manifold of constant sectional curvature, Chen [12] and Yau [39] have studied the surfaces with parallel mean curvature vector field in **X** and proved several remarkable results. However, for the complex counterpart, not many examples of the surfaces with parallel mean curvature vector field in $\mathbf{P}^n(\mathbf{C})$ have been known so far even when $n = 2$. We shall therefore focus our study on non-minimal surfaces in $\mathbf{P}^2(\mathbf{C})$ with parallel mean curvature vector field and investigate their existence and rigidity properties by means of the method similar to those developed in Chapter II.

The following theorem gives us a local existence of an immersion with parallel mean curvature vector field in a complex 2-dimensional Kaehler manifold.

Theorem (Theorem 12.1). *Let* b *and* ρ *be real numbers* ($b > 0$), *and* λ, α *and* a *be real-valued smooth functions of single variable* u *defined on an interval* I, *such that they satisfy the following system of ordinary differential equations:*

$$\frac{d\lambda}{du} = -\lambda^2 \cot(\alpha)(a-b),$$
$$\frac{d\alpha}{du} = \lambda(a+b),$$
$$\frac{da}{du} = \lambda\{2\cot(\alpha)(a-b)a + \frac{3}{4}\rho\sin(2\alpha)\}.$$

Let $\mathbf{M}$ *be an open domain of* (u,v)*-plane contaied in* $\mathrm{I}\times(-1,1)$*, and* $ds^2 = \lambda^2(du^2+dv^2)$ *be a Riemannian metric on* $\mathbf{M}$*. Then there exists an isometric immersion* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *of* $\mathbf{M}$ *into a 2-dimensional Kaehler manifold* $\mathbf{X}$ *of constant holomorphic sectional curvature* 4ρ *which satisfies the following:*

(1) *x has a non-zero parallel mean curvature vector field whose length is equal to* $2b$*,*

(2) *the Kaehler function of x is $\cos(\alpha)$ and*

(3) *the second fundamental form of x is explicitly written in terms of a,b, λ and α.*

It is remarked that when x is minimal, that is $b=0$, Eschenburg et al.[20] proved the corresponding local existence theorem of such immersions.

We define some local invariants which characterize immersions with parallel mean curvature vector field. For example, there exists a complex-valued smooth function c defined locally on $\mathbf{M}$ which satisfies

$$\frac{\partial(\lambda^2 c)}{\partial z} = 0,$$

where $\{z\}$ is a local isothermal coordinate such that the metric of $\mathbf{M}$ is $\mathrm{ds}^2 = \lambda^2|dz|^2$.

Next by using a function c stated above, we have the following theorem which gives a necessary and sufficient condition for the congruence of two non-minimal isometric immersions of $\mathbf{M}$ into $\mathbf{X}$ with parallel mean curvature vector field.

Theorem (Corollary 13.2). *Let* $\mathbf{M}$ *be a simply connected* 2*-dimensional Riemannian manifold and* $\mathbf{X}$ *a complex* 2*-dimensional Kaehler manifold of constant holomorphic sectional curvature* 4ρ. *Let* $x_i : \mathbf{M} \longrightarrow \mathbf{X}$ $(i = 1, 2)$ *be isometric immersions with non-zero parallel mean curvature vector field* $\mathbf{H}_i$ *and the Kaehler function* $cos(\alpha_i)$*, which are neither holomorphic nor anti-holomorphic. Assume that* x_1 *is isometric to* x_2. *Then* x_1 *is congruent to* x_2 *if and only if*

$$cos(\alpha_1) = cos(\alpha_2), \ \| \mathbf{H}_1 \| = \| \mathbf{H}_2 \| \ \text{and } c_1 = c_2.$$

Now we consider the case that $\mathbf{M}$ is of constant Gaussian curvature. Applying Theorem (Theorem 12.1) we know that x is totally real and $\mathbf{M}$ is flat. Hence, we have

Theorem (Proposition 14.1). *Let* $\mathbf{M}$ *be a simply connected* 2*-dimensional Riemannian manifold of constant Gaussian curvature* K *and* $x : \mathbf{M} \longrightarrow \mathbf{P}^2(\mathbf{C})$ *an isometric immersion such that the mean curvature vector field is parallel and not zero. Then* x *is totally real and* $\mathrm{K} = 0$.

Theorem (Proposition 14.1) gives a generalization of a result due to Kenmotsu [24,Theorem 3]. We can determine all isometric totally real immersions of a complete flat surface into $\mathbf{P}^2(\mathbf{C})$ with parallel mean curvature vector field. In consequence, we obtain

Theorem (Theorem 14.2). *Let* $\mathbf{R}^2$ *be the Euclidean* 2*-plane with the standard flat metric and* $x : \mathbf{R}^2 \longrightarrow \mathbf{P}^2(\mathbf{C})$ *an isometric immersion with parallel mean curvature vector field. Then* $x(\mathbf{R}^2)$ *is an orbit of the abelian Lie subgroup* G *of* $\mathrm{U}(3)$ *and* G *is algebraically determined by the constant* b*, where* $2b$ *is the length of the mean curvature vector field* $\mathbf{H}$ *of* x.

Theorem (Theorem 14.2) gives a generalization of a theorem by Ludden, Okumura and Yano [30].

This thesis is organized as follows. In Chapter I we discuss Simon's conjecture, which is closely related to the rigidity of generalized Veronese surfaces in $\mathbf{S}^n$. Chapter II is devoted to the rigidity problems of some minimal surfaces in $\mathbf{P}^n(\mathbf{C})$. In Chapter III we study non-minimal surfaces with parallel mean curvature vector field in a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature and discuss their local existence and rigidity properties.

Chapter I
Generalized Veronese Surfaces in S^n

1. Higher order osculating spaces

In this section we first introduce the concept of the m-th fundamental form of an isometric immersion and by using it we define some scalar fields on an immersed surface and study their properties.

Let $\bar{M}$ be an n-dimensional Riemannian manifold of constant sectional curvature c and $x : M \longrightarrow \bar{M}$ be an isometric minimal immersion of a 2-dimensional oriented Riemannian manifold M into $\bar{M}$. In this Chapter we shall agree on the following ranges of indices:

$$1 \leq A, B, \ldots \leq n,$$

$$1 \leq i, j, \ldots \leq 2; \quad 3 \leq \alpha, \beta, \ldots \leq n.$$

Let $\{e_A\}$ be a local orthonormal frame field on $\bar{M}$ and $\{\theta_B\}$ the coframe field dual to $\{e_A\}$. The structure equations of $\bar{M}$ are given by

$$\begin{aligned} d\theta_A &= \sum_B \theta_{AB} \wedge \theta_B, \quad \theta_{AB} + \theta_{BA} = 0, \\ d\theta_{AB} &= \sum_C \theta_{AC} \wedge \theta_{CB} + \Theta_{AB}, \\ \Theta_{AB} &= -c\theta_A \wedge \theta_B. \end{aligned} \tag{1.1}$$

To study the geometry of the immersed surface M, we consider an orthonormal frame field $\{e_A\}$ over M for which e_i are tangent to M and e_α are normal to M. Such a frame $\{e_i, e_\alpha\}$ is called an *adapted frame* along x. Then we have $\theta_\alpha = 0$. By the first equation of (1.1) and Cartan's lemma, we can put

$$\theta_{i\alpha} = \sum_j h_{\alpha ij}\theta_j, \quad \text{where} \quad h_{\alpha ij} = h_{\alpha ji}.$$

The vector-valued symmetric 2-form defined by

$$\sum h_{\alpha ij}\theta_i \otimes \theta_j \otimes e_\alpha$$

is called the *second fundamental form* of x. The condition that x is minimal is written as

$$\sum_i h_{\alpha ii} = 0.$$

We denote by K the Gaussian curvature of M. The Gauss equation is given as

$$\mathrm{K} = c - \sum_\alpha (h^2_{\alpha 11} + h^2_{\alpha 12}).$$

A curve C in M through a point $p \in \mathrm{M}$ is a smooth map $C(t)$ into M, $|t| < \mathrm{L}$, with $C(0) = p$. By covariant differentiation along C we get the vector fields

$$(1.2) \qquad \frac{DC}{dt}, \frac{D^2C}{dt^2}, \ldots, \frac{D^mC}{dt^m}, \ldots.$$

The first m vectors in (1.2) at $t = 0$ are said to span the osculating space of order m of $C(t)$ at $p = C(0)$. The *m-th osculating space* $T_p^{(m)}$ of M at $p \in \mathrm{M}$ is defined to be the space spanned by all the osculating spaces of order m at p of the curves through p and lying on M. We then have

$$T_p^{(1)} \subset T_p^{(2)} \subset \cdots \subset T_p^{(m)} \subset \cdots,$$

where $T_p^{(1)}$ is the tangent space $T_p(\mathrm{M})$ to M at p. We define $O_p^{(m)}$ to be the orthogonal complement of $T_p^{(m)}$ in $T_p^{(m+1)}$, so that we have

$$T_p^{(m+1)} = T_p^{(m)} \oplus O_p^{(m)}, \ m = 0, 1, \ldots.$$

$O_p^{(m)}$ is called the *m-th normal space*. Note that $T_p^{(0)} = \{0\}$ and $O_p^{(0)} = T_p\mathrm{M}$. We define $l_m(p) = \dim(O_p^{(m)})$. Then we have

$$(1.3) \qquad dim(T_p^{(m)}) = 2 + l_1(p) + \cdots + l_{m-1}(p), \quad m = 1, 2, \ldots.$$

A point $p \in \mathbf{M}$ is called a *regular point of order* m if $l_k(p) = 2, k = 1, \ldots, m-1$ in a neighbourhood of p. Let $\Omega_{(m)}$ be the open set of all regular points of order m. We set $\Omega_{(1)} = \mathbf{M}$. Then

$$\Omega_{(1)} \supset \Omega_{(2)} \supset \cdots \supset \Omega_{(m)}.$$

Suppose now that p is a regular point of order m. We denote by $N_p^{(m)}$ the orthoganal complement of $T_p^{(m)}$ in $T_p\bar{\mathbf{M}}$. In what follows, we shall use the following ranges of indices:

$$2k+1 \leq \lambda_k \leq 2k+2, \quad k = 0, 1, \ldots, m-1, \quad 2m+1 \leq \lambda_m \leq n.$$

Let $\{e_{\lambda_0}, e_{\lambda_1}, \ldots, e_{\lambda_m}\}$ be an adapted frame along x such that $\{e_{\lambda_0}, e_{\lambda_1}, \ldots, e_{\lambda_k}\}$ spans $T_p^{(k+1)}$ (k = 0,1,.., $m-1$) and hence $\{e_{\lambda_m}\}$ spans $N_p^{(m)}$. We then have

$$\theta_{\lambda_b\lambda_a} = 0 \quad \text{for } b = 0, 1, \ldots, m-2, \; a = (b+2), \ldots, m. \tag{1.4}$$

By taking the exterior derivative of (1.4) and making use of the structure equations, we get

$$\sum_{\lambda_{b+1}} \theta_{\lambda_b\lambda_{b+1}} \wedge \theta_{\lambda_{b+1}\lambda_{b+2}} = 0, \quad b = 0, \ldots, m-2. \tag{1.5}$$

From this we can define inductively the quantities $h_{\lambda_b i_1 \cdots i_{b+1}}$ in the following way.

$$\sum_{\lambda_b} h_{\lambda_b i_1 \cdots i_{b+1}} \theta_{\lambda_b\lambda_{b+1}} = \sum_{i_{b+2}} h_{\lambda_{b+1} i_1 \cdots i_{b+2}} \theta_{i_{b+2}}, \quad b = 0, 1, \ldots, m-1, \tag{1.6}$$

where we put $h_{\lambda_0 i} = 1$. The vector-valued symmetric (k+1)-form defined by

$$\sum h_{\lambda_b i_1 \cdots i_{b+1}} \theta_{i_1} \otimes \cdots \otimes \theta_{i_{b+1}} \otimes e_{\lambda_b}$$

is called the (b+1)-*th fundamental form* of x. This has the following properties:

(1.7)
(1) $h_{\lambda_b i_1 \cdots i_{b+1}}$ is symmetric in the set of indices $i_1, i_2, \ldots, i_{b+1}$,
(2) $\sum_i h_{\lambda_b i_1 \cdots i \cdots i \cdots i_{b+1}} = 0$,
(3) $< e_{\lambda_b}, D^{b+1}x >= \sum h_{\lambda_b i_1 \cdots i_{b+1}} \theta_{i_1} \otimes \theta_{i_{b+1}}$.

We now introduce the following notations for brevity:

$$1[k] = 1 \cdots 1(k\text{−times})$$

and put

$$V_1^{(b)} = \sum h_{\lambda_b 1[b]1} \ , \ V_2^{(b)} = \sum h_{\lambda_b 1[b]2}.$$

Then (1.7) means that the subspace $O_p^{(b)}$ is spanned by $\mathrm{V}_1^{(b)}$ and $\mathrm{V}_2^{(b)}$, since p is a regular point of order m.

On each $\Omega_{(b)}$ we introduce three smooth functions $\mathrm{K}_{(b+1)}$, $\mathrm{N}_{(b+1)}$ and $\mathrm{f}_{(b+1)}$, $b = 1,2,\ldots,m$, which are defined by

$$\mathrm{K}_{(b+1)} = \sum_{\lambda_b}(h^2_{\lambda_b 1[b]1} + h^2_{\lambda_b 1[b]2}),$$

$$\mathrm{N}_{(b+1)} = (\sum_{\lambda_b} h^2_{\lambda_b 1[b]1})(\sum_{\lambda_b} h^2_{\lambda_b 1[b]2}) - (\sum_{\lambda_b} h_{\lambda_b 1[b]1} h_{\lambda_b 1[b]2})^2, \tag{1.8}$$

$$\mathrm{f}_{(b+1)} = \mathrm{K}^2_{(b+1)} - 4\mathrm{N}_{(b+1)}.$$

We remark that these functions are independent of the choice of adapted frame fields and hence give rise to scalar invariants of x globally defined on $\Omega_{(b)}$ (cf. [22]). For the geometric meaning of these scalars, we have the following lemmas.

Lemma 1.1 (Ôtsuki [37]). *If* $\mathrm{M} = \Omega_{(b)}$, $\mathrm{N}_{(b)} \neq 0$ *and* $\mathrm{K}_{(b+1)} = 0$ *on* M, *then there exists a* $2b$*-dimensional totally geodesic submanifold of* $\bar{\mathrm{M}}$ *which contains* M.

Lemma 1.2 (Chern [16] and Kenmotsu [22]). *Let* M *be a compact, oriented, connected minimal surface in* S^n. *Suppose that* M *is not totally geodesic, the Gaussian curvature of* M *is strictly positive and* $\Omega_{(m)}$ *is not empty for a positive integer* m. *Then, for each* $b \leq m$, *we have* $\mathrm{f}_{(b+1)} = 0$ *on* $\Omega_{(b)}$ *and* $\mathrm{M}\backslash\Omega_{(b)}$ *are at most finite.*

Let $x : \mathrm{M} \longrightarrow \mathrm{S}^n$ be an isometric immersion of an oriented 2-dimensional Riemannian manifold M into S^n. It is well-known that the necessary and sufficient condition for x to be minimal is given by

$$\Delta x = -2x, \tag{1.9}$$

where Δ is the Laplacian for the metric on $\mathbf{M}$ (cf. [12] and [26]). Now we define generalized Veronese surfaces in $\mathbf{S}^n$, which are all minimal and of constant Gaussian curvature:

If $\mathbf{M}$ is a 2-sphere $\mathbf{S}^2(\mathrm{K})$ of constant Gaussian curvature K, then the eigenvalues of the Laplacian Δ are

$$\lambda_l = -l(l+1)\mathrm{K}, \quad l = 0, 1, 2, \ldots.$$

The eigenspace W_l corresponding to each λ_l consists of spherical harmonic polynomials of degree l, that is, the restriction to $\mathbf{S}^2(\mathrm{K})$ of harmonic homogeneous polynomials of degree l, which has dimension $(2l+1)$.

For a positive integer l, we put

$$\mathrm{K}(l) = \frac{2}{l(l+1)}. \tag{1.10}$$

If $\{f_0, \ldots, f_{2l}\}$ is an orthonormal basis for W_l with respect to the L^2-norm, then

$$\varphi_l : \mathbf{S}^2(\mathrm{K}(l)) \longrightarrow \mathbf{R}^{2l+1}$$

defined by

$$\varphi_l = \frac{1}{\sqrt{2l+1}}(f_0, f_1, \ldots, f_{2l})$$

gives rise to an isometric minimal immersion of $\mathbf{S}^2(\mathrm{K}(l))$ into $\mathbf{S}^{2l}$. For the rigidity of these minimal immersions $\{\varphi_l\}$, we have the following

Theorem 1.3 (Calabi [11]).

(1) *Let* $\mathbf{S}^2(\mathrm{K})$ *be a 2-sphere of constant Gaussian curvature* K *and* $x : \mathbf{S}^2(\mathrm{K}) \longrightarrow \mathbf{S}^n$ *a full isometric minimal immersion of* $\mathbf{S}^2(\mathrm{K})$ *into* $\mathbf{S}^n$. *Then there exists some*

integer k with $0 \leq k \leq n/2$ such that x is congruent to φ_k. Hence we have $\mathrm{K} = \mathrm{K}(k)$ *and* $n = 2k$.

(2) *For each integer k with $0 \leq k \leq n/2$, there exists an isometric minimal immersion*

$$\varphi_k : \mathbf{S}^2(\mathrm{K}(k)) \longrightarrow \mathbf{S}^n,$$

and any two such immersions φ_k and φ'_k differ by an isometry of $\mathbf{S}^n$.

For example, the immersion $\varphi_2 : \mathbf{S}^2(\mathrm{K}(2)) \longrightarrow \mathbf{S}^4$ is given by

$$\varphi_2(x, y, z) = \frac{1}{\sqrt{3}}(yz, zx, xy, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2)),$$

where $\mathbf{S}^2(\mathrm{K}) = \{(x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 = 1/\mathrm{K}\}$. φ_2 is called the *Veronese immersion* (or $\varphi_2(\mathbf{S}^2(1/3))$ is called the *Veronese surface* in $\mathbf{S}^4$). We remark that two points (x, y, z) and $(-x, -y, -z)$ of $\mathbf{S}^2(1/3)$ are mapped into the same point of $\mathbf{S}^4$ and so φ_2 defines in fact an imbedding of the real projective plane into $\mathbf{S}^4$. The immersion φ_l $(l \geq 3)$ stated above is a generalization of φ_2 and called the *generalized Veronese immersion* (or $\varphi_l(\mathbf{S}^2(\mathrm{K}(l))$ is called the *generalized Veronese surface* in $\mathbf{S}^{2l}$). The details can be found in [10], [11] and [19].

2. The second and third fundamental forms

Let $x : \mathbf{M} \longrightarrow \mathbf{S}^n$ be an isometric minimal immersion of an oriented 2-dimensional Riemannian manifold $\mathbf{M}$ into $\mathbf{S}^n$. In this section we shall calculate the Laplacians of the squares of the length of second and third fundamental forms $(\sum h^2_{\alpha ij})$ and $(\sum h^2_{\alpha ijk})$. Then, by the maximum principle, we obtain several estimates for these quantities, which enable us to consider the pinching problems concerning the Gaussian curvature K.

Suppose that $\mathbf{M}$ is complete and the Gaussian curvature K of $\mathbf{M}$ is bounded from below by some positive constant, which implies that $\mathbf{M}$ is compact. Let $\{e_i, e_\alpha\}$

be an adapted frame along $\mathbf{M}$. We define the covariant derivatives $\mathrm{h}_{\alpha ij,k}$ of $\mathrm{h}_{\alpha ij}$ by

$$Dh_{\alpha ij} = \sum_k h_{\alpha ij,k}\theta_k = dh_{\alpha ij} + \sum_\beta h_{\beta ij}\theta_{\beta\alpha} + \sum_s h_{\alpha sj}\theta_{si} + \sum_s h_{\alpha is}\theta_{sj}.$$

By the definition of $\mathrm{h}_{\alpha ij}$ together with the structure equations (1.1), we get $\mathrm{h}_{\alpha ij,k} = \mathrm{h}_{\alpha ik,j}$. Since x is minimal, this is equivalent to

$$h_{\alpha 11,2} = h_{\alpha 12,1}, \quad h_{\alpha 12,2} = -h_{\alpha 11,1}. \tag{2.1}$$

If x is totally geodesic in $\mathbf{S}^n$, then we have $\mathrm{K}_{(2)} = 0$ on $\mathbf{M}$.

Now we assume that x is not totally geodesic. By Lemma 1.2 we then have that $\Omega_{(2)} \neq \emptyset$ and $\mathrm{f}_{(2)} = 0$. Hence the normal vectors $\sum h_{\alpha 11}e_\alpha$ and $\sum h_{\alpha 12}e_\alpha$ are perpendicular to each other, which have the same non-zero length at any p in $\Omega_{(2)}$. From this it follows that

$$\Delta \mathrm{K}_{(2)} = 2\,\Delta\,(\sum h^2_{\alpha 11}),$$

where Δ is the Laplacian for the metric of $\mathbf{M}$. By $(4.27)_2$ of [22], we have

$$\Delta(\sum h^2_{\alpha 11}) = 4(\sum h^2_{\alpha 11})\mathrm{K} - 4(\sum h^2_{\alpha 11})^2 + 2\sum(h^2_{\alpha 11,1} + h^2_{\alpha 11,2}). \tag{2.2}$$

The first normal space $O^{(1)}_p$ is spanned by $\sum h_{\alpha 11}e_\alpha$ and $\sum h_{\alpha 12}e_\alpha$ by virtue of (3) in (1.7). Normalizing these vectors, we adopt them as a basis of $O^{(1)}_p$ for $p \in \Omega_{(2)}$. With respect to this new frame we then

$$h_{311} = h_{412}, \quad h_{312} = h_{411} = 0 \quad \text{and} \quad h_{\alpha ij} = 0 \; (\alpha \geq 5),$$

$$dh_{311} = h_{311,1}\theta_1 + h_{311,2}\theta_2,$$

$$h_{311}(-\theta_{34} + 2\theta_{12}) = h_{311,2}\theta_1 - h_{311,1}\theta_2, \tag{2.3}$$

$$h_{411,1} = -h_{311,2}, \; h_{411,2} = h_{311,1},$$

$$h_{311}\theta_{3\alpha} = h_{\alpha 111}\theta_1 + h_{\alpha 112}\theta_2,$$

$$h_{311}\theta_{4\alpha} = h_{\alpha 112}\theta_1 - h_{\alpha 111}\theta_2, \; (\alpha \geq 5).$$

The covariant derivatives $h_{\alpha ijk,l}$ of $h_{\alpha ijk}$ are defined by

$$Dh_{\alpha ijk} = \sum_l h_{\alpha ijk,l}\theta_l = dh_{\alpha ijk} + \sum h_{\beta ijk}\theta_{\beta\alpha} + \sum h_{\alpha sjk}\theta_{si} + \sum h_{\alpha isk}\theta_{sj} + \sum h_{\alpha ijs}\theta_{sk}.$$

Then we have

$$\sum_i h_{\alpha iik,l} = 0.$$

Taking the exterior derivative of the fifth equation of (2.3), we have

$$(2.4) \qquad h_{\alpha 111,2} = h_{\alpha 112,1} \quad \text{and} \quad h_{\alpha 111,1} + h_{\alpha 112,2} = 0.$$

Recall that, since x is not totally geodesic and $f_{(2)} = 0$ on M, we know $\Omega_{(2)} \neq \emptyset$. Hence by Lemma 1.2, we get $f_{(3)} = 0$ on $\Omega_{(2)}$. From this it follows that $V_1^{(2)}$ and $V_2^{(2)}$ are perpendicular to each other and have the same non-zero length on $\Omega_{(2)}$, from which we have $\sum h_{\alpha 111}^2 = \sum h_{\alpha 112}^2$ and $\sum h_{\alpha 111}h_{\alpha 112} = 0$.

Lemma 2.1. *On $\Omega_{(2)}$, we have*

$$\Delta(\sum h_{\alpha 111}^2) = 6(\sum h_{\alpha 111}^2)\mathrm{K} - \frac{4}{h_{311}^2}(\sum h_{\alpha 111}^2)^2 + 2\sum(h_{\alpha 111,1}^2 + h_{\alpha 111,2}^2).$$

Proof. We first get

$$d(\sum h_{\alpha 111}^2) = 2\sum(h_{\alpha 111}h_{\alpha 111,1}\theta_1 + h_{\alpha 111}h_{\alpha 111,2}\theta_2),$$

$$\Delta(\sum h_{\alpha 111}^2)\theta_1 \wedge \theta_2 = 2d\{\sum(h_{\alpha 111}h_{\alpha 111,1}\theta_2 - h_{\alpha 111}h_{\alpha 111,2}\theta_1)\}.$$

On the other hand, by (2.4), we have

$$2\sum(h_{\alpha 111}h_{\alpha 111,1}\theta_2 - h_{\alpha 111}h_{\alpha 111,2}\theta_1)$$
$$= -2\sum h_{\alpha 111}(dh_{\alpha 112} + \sum h_{\beta 112}\theta_{\beta\alpha} + 3h_{\alpha 111}\theta_{12}).$$

By a direct calculation we have

$$\Delta(\sum h_{\alpha 111}^2)\theta_1 \wedge \theta_2 = 6\mathrm{K}\sum h_{\alpha 111}^2\theta_1 \wedge \theta_2 - 2\sum h_{\alpha 111}h_{\beta 112}\theta_{\beta 3} \wedge \theta_{3\alpha}$$
$$-2\sum h_{\alpha 111}h_{\beta 112}\theta_{\beta 4} \wedge \theta_{4\alpha} + 2(\sum h_{\alpha 111,1}^2 + h_{\alpha 111,2}^2)\theta_1 \wedge \theta_2.$$

Substituting the fifth and sixth equations of (2.3) into the above equation, we obtain Lemma 2.1. q.e.d.

By making use of (2.2) and the first equation of (2.3), we have

$$\Delta h_{311}^2 = 4h_{311}^2 \mathrm{K} - 4h_{311}^4 + 4(h_{311,1}^2 + h_{311,2}^2) + 4\sum h_{\alpha 111}^2. \tag{2.5}$$

Then, by Lemma 2.1 and (2.5), we get the Laplacian of a smooth function $(h_{311}^2 \sum h_{\alpha 111}^2)$ defined on $\Omega_{(2)}$ as follows:

$$\begin{aligned}\Delta(h_{311}^2 \textstyle\sum h_{\alpha 111}^2) &= 10h_{311}^2 \mathrm{K}(\textstyle\sum h_{\alpha 111}^2) - 4h_{311}^4(\textstyle\sum h_{\alpha 111}^2)\\ &\quad + 2h_{311}^2 \textstyle\sum(h_{\alpha 111,1}^2 + h_{\alpha 111,2}^2) + 4\textstyle\sum h_{\alpha 111}^2(h_{311,1}^2 + h_{311,2}^2)\\ &\quad + 8(h_{311}h_{311,1}\textstyle\sum h_{\alpha 111}h_{\alpha 111,1} + h_{311}h_{311,2}\textstyle\sum h_{\alpha 111}h_{\alpha 111,2}).\end{aligned} \tag{2.6}$$

Taking the exterior derivative of $\sum h_{\alpha 111}^2 = \sum h_{\alpha 112}^2$ and $\sum h_{\alpha 111}h_{\alpha 112} = 0$, we have

$$\begin{aligned}&\textstyle\sum(h_{\alpha 111}h_{\alpha 111,2} + h_{\alpha 112}h_{\alpha 111,1}) = 0,\\ &\textstyle\sum(h_{\alpha 111}h_{\alpha 111,1} - h_{\alpha 112}h_{\alpha 111,2}) = 0,\end{aligned} \tag{2.7}$$

from which (2.6) implies

$$\begin{aligned}\Delta(h_{311}^2 \textstyle\sum h_{\alpha 111}^2) &= 2h_{311}^2 \textstyle\sum h_{\alpha 111}^2(5 - 12h_{311}^2)\\ &\quad + 2\textstyle\sum\{(h_{311}h_{\alpha 111,1} + h_{311,1}h_{\alpha 111} - h_{311,2}h_{\alpha 112})^2\\ &\quad + (h_{311}h_{\alpha 111,2} + h_{311,1}h_{\alpha 112} - h_{311,2}h_{\alpha 111})^2\}.\end{aligned} \tag{2.8}$$

This formula (2.8) will be used essentially in the proof of the following Theorem 2.2.

Now we consider the characterization of the generalized Veronese immersion φ_l in $\mathbf{S}^n$ in terms of the pinched Gaussian curvature. By (1.10) we see $\mathrm{K}(l) = 1$, $1/3$ and $1/6$ for $l = 1$, 2 and 3, respectively. It is then an interesting problem to classify all minimal surfaces in $\mathbf{S}^n$ for which the Gaussian curvature K satisfies the following conditions:

(2.9) $\qquad$ $(1)\ 1 \geq \mathrm{K} \geq 1/3, (2)\ 1/3 \geq \mathrm{K} \geq 1/6, (3)\ 1 \geq \mathrm{K} \geq 1/6, (4)\ 1/6 \geq \mathrm{K} > 0.$

The following Theorem 2.2 gives the answer to the case (3) in (2.9).

Theorem 2.2. *Let $x : \mathbf{M} \longrightarrow \mathbf{S}^n$ be an isometric minimal immersion of a complete, connected, oriented 2-dimensional Riemannian manifold $\mathbf{M}$ into $\mathbf{S}^n$ ($n \geq 3$). If the Gaussian curvature K of $\mathbf{M}$ satisfies*

$$\frac{1}{6} \leq \mathrm{K} \leq 1,$$

then $x(\mathbf{M})$ is either

(1) *a totally geodesic surface in $\mathbf{S}^4$ and $\mathrm{K} \equiv 1$,*

(2) *the generalized Veronese surface in $\mathbf{S}^6$ and $\mathrm{K} \equiv 1/6$, or*

(3) *a minimal surface in $\mathbf{S}^4$ with $1/6 \leq \mathrm{K} \leq 1$.*

Proof. Since the condition $\mathrm{K} \geq 1/6$ implies $\mathrm{h}_{311}^2 \leq 5/12$ on $\mathbf{M}$, the formula (2.8) shows

$$\Delta(h_{311}^2 \sum h_{\alpha 111}^2) \geq 0 \quad \text{on} \quad \Omega_{(2)}.$$

By Lemma 1.2, $\mathbf{M}\backslash\Omega_{(2)}$ is at most finite. Hence $\Omega_{(2)}$ is parabolic or compact, and the maximum principle of subharmonic functions holds on $\Omega_{(2)}$. If there exists a point p of $\mathbf{M}\backslash\Omega_{(2)}$ such that

$$\limsup_{x\to p}(h_{311}^2 \sum h_{\alpha 111}^2)(x) = +\infty,$$

then we have

$$\limsup_{x\to p}(\sum h_{\alpha 111}^2)(x) = +\infty.$$

Hence, by (2.5), it follows that

$$\limsup_{x\to p}(\Delta h_{311}^2) \geq \limsup_{x\to p}(4 \sum h_{\alpha 111}^2) = +\infty,$$

since $\limsup_{x\to p} h_{311}^2 = 0$. But this contradicts the boundedness of $\triangle h_{311}^2$ on the compact manifold $\mathbf{M}$. Hence $h_{311}^2 \sum h_{\alpha 111}^2$ is bounded from above and subharmonic on the parabolic surface $\Omega_{(2)}$, so is constant on $\Omega_{(2)}$. Thus, from (2.8) we have $(5-12h_{311}^2) \equiv 0$ or $\sum h_{\alpha 111}^2 \equiv 0$. If $(5-12h_{311}^2) \equiv 0$, we have $\mathbf{M} = \Omega_{(2)}$, $\mathrm{K} \equiv 1/6$ and $\sum h_{\alpha 111}^2 = 5/48$ by (2.5). Moreover, by Lemma 2.1, we have $\sum(h_{\alpha 111,1}^2 + h_{\alpha 111,2}^2) = 0$, which shows $n = 6$ by Lemma 1.1. On the other hand, if $\sum h_{\alpha 111}^2 = 0$, we have $n = 4$ by Lemma 1.1. q.e.d.

The case (1) in (2.9) has been proved by Calabi [11], Lawson [28] (for n=4) , Benko et.al. [4] and Kozlowski and Simon [27] (for any n). The case (2) in (2.9) has been proved by Kozlowski and Simon [27] and Ogata [32], independently, as follows:

Corollary 2.3. *Under the same assumptions as in Theorem* 2.2, *if the Gaussian curvature* K *satisfies*

$$\frac{1}{6} \leq \mathrm{K} \leq \frac{1}{3},$$

then $\mathrm{K} \equiv 1/3$ *or* $1/6$. *Moreover,* $x(\mathbf{M})$ *is either the Veronese surface in* $\mathbf{S}^4$ *if* $\mathrm{K} \equiv 1/3$, *or the generalized Veronese surface in* $\mathbf{S}^6$ *if* $\mathrm{K} \equiv 1/6$.

Proof. To prove Corollary 2.3, it suffices to show that a minimal surface in $\mathbf{S}^4$ satisfying $1/6 \leq \mathrm{K} \leq 1/3$ is the Veronese surface. From (2.5), we have

$$\triangle h_{311}^2 = 4h_{311}^2(1 - 3h_{311}^2) + 4(h_{311,1}^2 + h_{311,2}^2).$$

On the other hand, $1/6 \leq \mathrm{K} \leq 1/3$ implies $1/3 \leq h_{311}^2 \leq 5/12$ by the Gauss equation. Then we have

$$\triangle \log(1/h_{311}^2) = 4(3h_{311}^2 - 1) \geq 0.$$

Since $1/h_{311}^2$ is a positive scalar function on $\mathbf{M}$, we have $h_{311}^2 = 1/3$, which implies $\mathrm{K} = 1/3$. q.e.d.

We remark that the results of case (1) and (2) in (2.9) show that Simon's conjecture, which we shall study in the next Section 3, is true in these special cases.

3. Simon's conjecture

In 1980, Simon [38] has proposed the following conjecture, which is closely related to the rigidity properties of the generalized Veronese immersions.

Conjecture. *Let* $x : \mathbf{M} \longrightarrow \mathbf{S}^n$ *be a full isometric minimal immersion of a closed connected oriented* 2*-dimensional Riemannian manifold* $\mathbf{M}$ *into* $\mathbf{S}^n$*. If the Gaussian curvature* K *of* $\mathbf{M}$ *satisfies*

$$\mathrm{K}(k+1) \leq \mathrm{K} \leq \mathrm{K}(k)$$

for some positive integer k*, then either* $\mathrm{K} = \mathrm{K}(k)$ *or* $\mathrm{K} = \mathrm{K}(k+1)$*, and* x *is congruent to* φ_k *or* φ_{k+1}*, respectively.*

The case when $k = 1$ or 2 corresponds to the case (1) or (2) stated in Section 2. In this section we shall give an affirmative answer to Simon's conjecture under some higher order regularity assumption of x.

For this purpose, we calculate the $(m+1)$-th fundamental form and the $(m+1)$-th normal vectors. Let $x \in \mathbf{M}$ be a regular point of order $m(\geq 3)$. By Lemma 1.2 we have $\mathrm{f}_{(b)} = 0$ for $b = 1, 2, \ldots, m-1$. Hence the vectors $\mathrm{V}_1^{(b)}$ and $\mathrm{V}_2^{(b)}$ are perpendicular to each other and have the same non-zero length. Normalizing these vectors we adopt them as a basis of $O_p^{(b)}$. Since $\mathrm{V}_1^{(b)}$ and $\mathrm{V}_2^{(b)}$ are independent of the choice of the frame field, we have, with respect to this new frame on $\Omega_{(b+1)}$, for $b = 1, 2, \ldots, m-1$,

$$
\begin{aligned}
&h_{(2b+1)1[b]1} = h_{(2b+2)1[b]2} \ (\neq 0),\\
&h_{(2b+2)1[b]1} = h_{(2b+1)1[b]2} = 0,\\
&dh_{(2b+1)1[b]1} = h_{(2b+1)1[b]1,1}\theta_1 + h_{(2b+1)1[b]1,2}\theta_2,\\
&(b+1)h_{(2b+1)1[b]1}\theta_{12} - h_{(2b+1)1[b]1}\theta_{(2b+1)(2b+2)}\\
&\qquad = h_{(2b+1)1[b]1,2}\theta_1 - h_{(2b+1)1[b]1,1}\theta_2,
\end{aligned} \tag{3.1}
$$

$$h_{(2b+2)1[b]1,1} = -h_{(2b+1)1[b]1,2},$$

$$h_{(2b+2)1[b]1,2} = h_{(2b+1)1[b]1,1}.$$

By (1.6) and (3.1), we have, for $b = 1,2, \ldots, m-2$,

$$h_{(2b+1)1[b]1}\theta_{(2b+1)(2b+3)} = h_{(2b+3)1[b+1]1}\theta_1,$$

$$h_{(2b+1)1[b]1}\theta_{(2b+1)(2b+4)} = h_{(2b+4)1[b+1]2}\theta_2, \tag{3.2}$$

$$h_{(2b+2)1[b]2}\theta_{(2b+2)(2b+3)} = -h_{(2b+3)1[b+1]1}\theta_2,$$

$$h_{(2b+2)1[b]2}\theta_{(2b+2)(2b+4)} = h_{(2b+4)1[b+1]2}\theta_1.$$

By making use of the second, third and forth equations of (3.1) and (3.2), we have for $b = 1,2, \ldots, m-1$,

$$\Delta(h^2_{(2b+1)1[b]1}) = 2(b+1)\mathrm{K}h^2_{(2b+1)1[b]1} - 4h^2_{(2b+1)1[b]1}/h^2_{(2b-1)1[b-1]1} + 4(h^2_{(2b+1)1[b]1,1} + h^2_{(2b+1)1[b]1,2}) + 4h^2_{(2b+3)1[b+1]1}, \tag{3.3}$$

where we put

$$4h^2_{(2m+1)1[m]1} = 2\sum_{\lambda_m}(h^2_{\lambda_m 1[m-1]1,1} + h^2_{\lambda_m 1[m-1]1,2}).$$

By (1.6) the $(m+1)$-th fundamental tensors of x are defined by

$$\sum_{i_{m+1}} h_{\alpha i_1 \cdots i_{m+1}}\theta_{i_{m+1}} = \sum_{\lambda_{m-1}} h_{\lambda_{m-1} i_1 \cdots i_m}\theta_{\lambda_{m-1}\alpha}, \tag{3.4}$$

where we use the following range of indices: $2m+1 \leq \alpha, \beta, \ldots \leq n$.

We define the covariant derivatives $h_{\alpha i_1 \cdots i_{m+1},i}$ of $h_{\alpha i_1 \cdots i_{m+1}}$ by

$$Dh_{\alpha i_1 \cdots i_{m+1}} = \sum_i h_{\alpha i_1 \cdots i_{m+1},i}\theta_i = dh_{\alpha i_1 \cdots i_{m+1}} + \sum_\beta h_{\beta i_1 \cdots i_{m+1}}\theta_{\beta\alpha} + \sum_j h_{\alpha j i_2 \cdots i_{m+1}}\theta_{j i_1} + \cdots + \sum_j h_{\alpha i_1 \cdots j}\theta_{j i_{m+1}}.$$

We remark $\sum_i h_{\alpha i_1 \cdots i \cdots i \cdots i_{m+1},k} = 0$ by the formula (2) of (1.7). Taking the exterior derivative of (3.4), we have

$$h_{\alpha 1[m]1,2} = h_{\alpha 1[m]2,1} \quad \text{and} \quad h_{\alpha 1[m]1,1} + h_{\alpha 1[m]2,2} = 0. \tag{3.5}$$

Now we assume that $x \in \mathbf{M}$ is a regular point of order $(m+1)$. Then, by Lemma 1.2, we have $\mathrm{f}_{(m+1)} = 0$. Hence the $(m+1)$-th normal vectors $\mathrm{V}_1^{(m+1)}$ and $\mathrm{V}_2^{(m+1)}$ are perpendicular to each other and have the same non-zero length.

Lemma 3.1. *On $\Omega_{(m+1)}$, we have*

$$\Delta(\sum_\alpha h^2_{\alpha 1[m]1}) = 2(m+1)\mathrm{K}(\sum_\alpha h^2_{\alpha 1[m]1}) - 4\frac{(\sum_\alpha h^2_{\alpha 1[m]1})^2}{h^2_{(2m-1)1[m-1]1}}$$
$$+2\sum_\alpha (h^2_{\alpha 1[m]1,1} + h^2_{\alpha 1[m]1,2}).$$

Proof. We first have

$$d(\sum h^2_{\alpha 1[m]1}) = 2\sum (h_{\alpha 1[m]1} h_{\alpha 1[m]1,1}\theta_1 + h_{\alpha 1[m]1} h_{\alpha 1[m]1,2}\theta_2),$$

$$\Delta(\sum h^2_{\alpha 1[m]1})\theta_1 \wedge \theta_2 = 2d\{\sum h_{\alpha 1[m]1}(h_{\alpha 1[m]1,1}\theta_2 - h_{\alpha 1[m]1,2}\theta_1)\}.$$

Making use of (3.1), (3.2), (3.3) and (3.4) and calculating the above equation, we have Lemma 3.1. q.e.d.

Normalizing $\mathrm{V}_1^{(m+1)}$ and $\mathrm{V}_2^{(m+1)}$, we adopt them as a basis of $O_p^{(m+1)}$. Since $\mathrm{V}_1^{(m+1)}$ and $\mathrm{V}_2^{(m+1)}$ are independent of the choice of the frame field, we have

$$h_{(2m+1)1[m]1} = h_{(2m+2)1[m]2} (\neq 0),$$

$$h_{(2m+2)1[m]1} = h_{(2m+1)1[m]2} = 0,$$

$$dh_{(2m+1)1[m]1} = h_{(2m+1)1[m]1,1}\theta_1 + h_{(2m+1)1[m]1,2}\theta_2, \tag{3.6}$$

$$(m+1)h_{(2m+1)1[m]1}\theta_{12} - h_{(2m+1)1[m]1}\theta_{(2m+1)(2m+2)}$$
$$= h_{(2m+1)1[m]1,2}\theta_1 - h_{(2m+1)1[m]1,1}\theta_2,$$

$$h_{(2m+2)1[m]1,1} = -h_{(2m+1)1[m]1,2},$$
$$h_{(2m+2)1[m]1,2} = h_{(2m+1)1[m]1,1}.$$

Substituting the first equation of (3.6) into (3.4), we have

$$\begin{aligned} h_{(2m-1)1[m-1]1}\theta_{(2m-1)(2m+1)} &= h_{(2m+1)1[m]1}\theta_1, \\ h_{(2m-1)1[m-1]1}\theta_{(2m-1)(2m+2)} &= h_{(2m+2)1[m]2}\theta_2, \\ h_{(2m)1[m-1]2}\theta_{(2m)(2m+1)} &= -h_{(2m+1)1[m]1}\theta_2, \\ h_{(2m)1[m-1]2}\theta_{(2m)(2m+2)} &= h_{(2m+2)1[m]2}\theta_1. \end{aligned} \tag{3.7}$$

The formulas (3.6) and (3.7) combined with Lemma 3.1 then imply that (3.1), (3.2) and (3.3) are valid for $b = m$. Hence, if (3.1), (3.2) and (3.3) hold for $b = 1,2, \ldots, m-1$ at a neighbourhood of a regular point of order m, then these formulas also hold for $b = 1,2,\ldots, m$ at a neighbourhood of order $(m+1)$. We remark that, for the case of $m = 2$, we have (2.3), (2.2) and Lemma 2.2 corresponding to (3.1) and (3.2), (3.3) and Lemma 3.1, respectively.

We define a smooth function $H_{(k-1)}$ on $\Omega_{(k)}$ by

$$H_{(k-1)} = h^2_{311}h^2_{5111}\cdots h^2_{(2k-1)1[k-1]1}, \tag{3.8}$$

which is defined globally on the components of $\Omega_{(k)}$ and is in fact a scalar invariant of the isometric minimal immersion x of M (cf. [22], pp. 474–475). We have

$$dH_{(k-1)} = 2H_{(k-1)}\{\frac{\sum_{b=1}^{k-1} h_{(2b+1)1[b]1,1}}{h_{(2b+1)1[b]1}}\theta_1 + \frac{\sum_{b=1}^{k-1} h_{(2b+1)1[b]1,2}}{h_{(2b+1)1[b]1}}\theta_2\}. \tag{3.9}$$

Using (3.3), Lemma 3.1 and (3.9) we have

$$\begin{aligned} \Delta H_{(k-1)} = 2H_{(k-1)}\{\frac{k(k+1)\mathrm{K}}{2} - 1\} + \frac{\mid dH_{(k-1)}\mid^2}{H_{(k-1)}} \\ + 4H_{(k-1)}\frac{\sum_{\lambda_k} h^2_{\lambda_k 1[k]1}}{h^2_{(2k-1)1[k-1]1}}. \end{aligned} \tag{3.10}$$

The following Lemma 3.2 then illustrates the fullness of immersions in question.

Lemma 3.2 (Calabi [11]). *Let $x : \mathrm{M} \longrightarrow \mathrm{S}^{2k}$ be a full isometric minimal immersion of a complete 2-dimensional Riemannian manifold* $\mathbf{M}$ *into* S^{2k} $(k \geq 2)$. *If the Gaussian curvature* K *of* $\mathbf{M}$ *is strictly positive, that is,* K *satisfies* $\delta \leq \mathrm{K}$ *for some positive number* δ, *then there exists a non-empty open set* U *of* $\mathbf{M}$ *such that* $\mathrm{K} \leq 2/k(k+1)$ *on* U.

We are now in a position to prove the following Theorems 3.3 and 3.4, which characterize the generalized Veronese immersion φ_k in terms of the fullness of an immersion and the properties of the pinched Gaussian curvature.

Theorem 3.3. *Let $x : \mathrm{M} \longrightarrow \mathrm{S}^n$ be a full isometric minimal immersion of a complete, connected, oriented 2-dimensional Riemannian manifold* $\mathbf{M}$ *into* S^n $(n \geq 3)$. *If the Gaussian curvature* K *of* $\mathbf{M}$ *satisfies*

$$\mathrm{K}(k) \leq \mathrm{K} \leq 1,$$

then either

(1) $x = \varphi_k$ *and* $\mathrm{K} = \mathrm{K}(k)$, *or*

(2) $n < 2k$.

Proof. By induction on k, we may assume that $\mathrm{M} = \Omega_{(k)}$ and $H_{(k-1)} \neq 0$. By making use of Lemma 1.1 and Lemma 1.2, we put $n = 2m$ for $m \geq k$. Hence, by Lemma 3.2, there exists a non-empty open set U such that $\mathrm{K} \leq 2/m(m+1)$ on U. It follows from the assumption of K and the connectedness of $\mathbf{M}$ that $m = k$ and U $= \mathbf{M}$. q.e.d.

Theorem 3.4. *Let $x : \mathrm{M} \longrightarrow \mathrm{S}^n$ be a full isometric minimal immersion of a complete, connected, oriented 2-dimensional Riemannian manifold* $\mathbf{M}$ *into*

$\mathbf{S}^n$ $(n \geq 3)$. *If the Gaussian curvature* K *of* **M** *satisfies*

$$\mathrm{K}(k) \geq \mathrm{K} \geq \delta$$

for some positive number δ, *and if each point of* **M** *is a regular point of order* k, *then either*

(1) $x = \varphi_k$ *and* $\mathrm{K} = \mathrm{K}(k)$, *or*

(2) $n > 2k$.

Proof. Suppose that $n \leq 2k$. By assumption we have $\Omega_{(k)} = \mathbf{M}$. Then, using (3.10), we have

$$\triangle \log H_{(k-1)} = 2\{\frac{k(k+1)}{2}\mathrm{K} - 1\}. \tag{3.11}$$

Hence $\log H_{(k-1)}$ is a bounded superharmonic function on **M**. This shows that $H_{(k-1)}$ is constant on **M** and $\mathrm{K} = 2/k(k+1)$. q.e.d.

Combining Theorem 3.3 with Theorem 3.4 we obtain the following theorem.

Theorem 3.5. *Let* $x : \mathbf{M} \longrightarrow \mathbf{S}^n$ *be a full isometric minimal immersion of a complete, connected, oriented 2-dimensional Riemannian manifold* **M** *into* $\mathbf{S}^n$ $(n \geq 3)$. *If the Gaussian curvature* K *of* **M** *satisfies*

$$\mathrm{K}(k+1) \leq \mathrm{K} \leq \mathrm{K}(k),$$

and if each point of **M** *is a regular point of order* k, *then either*

(1) $x = \varphi_k$ *and* $\mathrm{K} = \mathrm{K}(k)$, *or*

(2) $x = \varphi_{k+1}$ *and* $\mathrm{K} = \mathrm{K}(k+1)$.

By Theorem 3.5 we get an affirmative answer that Simon's conjecture is true if each point of **M** is a regular point of order k. It should be remarked that, if **M** does not have the regularity property, then the dimension of each osculating space is not constant and we can not use the formula (3.11).

4. Examples with $1 \geq \mathrm{K} \geq 1/6$

In this section we shall construct minimal surfaces in $\mathbf{S}^4$ with Gaussian curvature $\mathrm{K} \geq 1/6$, which give examples of the case (3) in Theorem 2.2.

Let z be an isothermal coordinate on $\mathbf{S}^2$. We define a one-parameter family $\{x_t;\, t \in (0, \infty)\}$ of immersions of $\mathbf{S}^2$ into $\mathbf{S}^4$, by

$$(4.1) \qquad x_t = \frac{1}{(t + 3|z|^2 + 3t^2|z|^4 + t|z|^6)} \begin{pmatrix} \sqrt{3t}(z^2 + \bar{z}^2)(|z|^2 + t) \\ -i\sqrt{3t}(z^2 - \bar{z}^2)(|z|^2 + t) \\ -i\sqrt{3t}(z - \bar{z})(t|z|^4 - 1) \\ \sqrt{3t}(z + \bar{z})(t|z|^4 - 1) \\ -t + 3|z|^2 + 3t^2|z|^4 - t|z|^6 \end{pmatrix}.$$

By a direct calculation, we have for each $t \in (0, \infty)$

$$(4.2) \qquad ds_t^2 = \frac{12t(1 + 4t^2|z|^2 + 6t|z|^4 + 4|z|^6 + t^2|z|^8)}{(t + 3|z|^2 + 3t^2|z|^4 + t|z|^6)^2} dz \otimes d\bar{z},$$

$$(4.3) \qquad \mathrm{K}_{(t)} = 1 - \frac{2(t + 3|z|^2 + 3t^2|z|^4 + t|z|^6)^4}{3t(1 + 4t^2|z|^2 + 6t|z|^4 + 4|z|^6 + t^2|z|^8)^3},$$

$$(4.4) \qquad \Delta_{(t)} x_t = -2x_t,$$

where ds_t^2 is the Riemannian metric of $\mathbf{S}^2$ induced by x_t and K_t (resp. $\Delta_{(t)}$) denote the Gaussian curvature (resp. the Laplacian) with respect to ds_t^2. From (4.4), we see that each immersion x_t is minimal. It is also verified that $x_1(\mathbf{S}^2)$ is the Veronese surface, and for each t > 0, x_t is not totally geodesic for $\mathrm{K}_{(t)} \neq 1$.

It is well-known that minimal immersions of 2-sphere $\mathbf{S}^2$ into $\mathbf{S}^n$ are closely related to holomorphic curves in $\mathbf{P}^n(\mathbf{C})$ (called the *directrix curves*) (cf. Barbosa [3], Calabi [11], Chern [14] and [15]). In fact, let $\mathbf{M}$ be a 2-dimensional Riemannian manifold and $x : \mathrm{M} \longrightarrow \mathbf{S}^n$ be an immersion of $\mathbf{M}$ into $\mathbf{S}^n$. Let $\pi : \mathbf{S}^n \longrightarrow \mathbf{P}^n(\mathbf{R})$ be the natural covering and $i : \mathbf{P}^n(\mathbf{R}) \longrightarrow \mathbf{P}^n(\mathbf{C})$ denote the natural inclusion, which is a totally real and totally geodesic imbedding. We consider $\varphi = i o \pi o x$: $\mathbf{M} \longrightarrow \mathbf{P}^n(\mathbf{C})$. Then it follows that φ is minimal if and only if x is minimal. Thus minimal immersions into $\mathbf{S}^n$ may be regarded as special cases of minimal immersions into $\mathbf{P}^n(\mathbf{C})$. The following Theorem 4.1 now gives us examples with the properties stated above.

Theorem 4.1. *The example* (4.1) *corresponds to the one-parameter family* ξ_t *of directrix curves in* $\mathbf{P}^4(\mathbf{C})$ *defined by Chern* ([14] *and* [15]) *which is given in homogeneous coordinate* :

$$\xi_t = \begin{pmatrix} 1 + tz^4 \\ i(1 - tz^4) \\ 2i(tz + z^3) \\ 2(-tz + z^3) \\ -2\sqrt{3t}z^2 \end{pmatrix}. \tag{4.5}$$

Proof. In $\mathbf{C}^5$ the symmetric product of two vectors $\mathbf{a} = (\mathrm{a}_i)$ and $\mathbf{b} = (\mathrm{b}_i)$ is given by

$$(\mathbf{a}, \mathbf{b}) = \sum_i a_i b_i.$$

Following Barbosa [3], we compute

$$G_2 = \bar{\partial}^2 x_t - \frac{(\bar{\partial}^2 x_t, \partial x_t)}{(\bar{\partial} x_t, \partial x_t)} \bar{\partial} x_t.$$

We then have $\xi_t = G_2/(G_2, \bar{G_2})$, which proves Theorem 4.1 by Theorem (3.30) of [3].

q.e.d.

In (4.3) we put $\mathrm{K}_{(t)} = 1 - \mathrm{L}_{(t)}$. Then we have

$$\mathrm{L}_{(t)} = \frac{2}{3t}\{t^4 + (1 - t^3) f\}, \text{ or} \tag{4.6}$$

$$\mathrm{L}_{(t)} = \frac{2}{3t}\{\frac{1}{t^2 + (1 - t^3) g}\}, \tag{4.7}$$

where $f = f(t, |z|^2)$ and $g = g(t, |z|^2)$ are some positive functions of t (>0) and $|z|^2$. If $1 \leq t^3 \leq 5/4$, then $\mathrm{L}_{(t)} \leq (2/3)t^3 \leq 5/6$ by (4.6), which implies $\mathrm{K}_{(t)} \geq 1/6$. Similarly, if $4/5 \leq t^3 \leq 1$, we have $\mathrm{K}_{(t)} \geq 1/6$ by (4.7). Thus we have $\mathrm{K}_{(t)} \geq 1/6$ for each t with $4/5 \leq t^3 \leq 5/4$, which gives examples of the case (3) in Theorem 2.2.

Remarks. (1). Corresponding to Tjaden's example in [27], we have the following one-parameter family $\tilde{\xi}_t$ of directrix curves in $\mathbf{P}^4(\mathbf{C})$:

$$\tilde{\xi}_t = \begin{pmatrix} (e^t + e^{-t}z^4) \\ i(e^t - e^{-t}z^4) \\ 2(z - z^3) \\ -2i(z + z^3) \\ 2\sqrt{3}z^2 \end{pmatrix}.$$

It is easily verified that ξ_t is isometric to some $\tilde{\xi}_{t'}$. Thus our example is the same as Tjaden's by [3, Proposition (5.2) and Theorem (5.15)].

(2). For any $\varepsilon > 0$ we set $t^3 = 1 + (3/2)\varepsilon$ (> 1). By (4.6) we have $\mathrm{L}_{(t)} \leq 2/3 + \varepsilon$, which implies $\mathrm{K}_{(t)} \geq 1/3 - \varepsilon$. This shows that the assumption on K in the case of $k = 1$ of Theorem 3.5 is best possible.

Chapter II

Curvature pinching theorems of minimal surfaces in $\mathbf{P}^n(\mathbf{C})$

5. Kaehler manifolds and J-canonical frames

In Chapter I we studied the Gaussian curvature pinching problems of minimal surfaces in $\mathbf{S}^n$ and proved several characterizing theorems of the generalized Veronese surfaces, which gave an affirmative answer to Simon's conjecture (cf. Section 3). In Chapter II we take the complex projective space $\mathbf{P}^n(\mathbf{C})$ as the ambient space and consider the Gaussian curvature pinching problems of minimal surfaces in $\mathbf{P}^n(\mathbf{C})$. As a result we prove the rigidity theorems for minimal surfaces in $\mathbf{P}^n(\mathbf{C})$. To start with, we study the properties of orthonormal frames of two kinds, defined generally in a Kaehler manifold, and their relations.

Let $\mathbf{X}$ be a Kaehler manifold of complex dimension n of constant holomorphic sectional curvature 4ρ, and $\{\omega_\alpha\}$ be a local field of unitary coframes on $\mathbf{X}$ so that the metric is represented by $ds^2 = \sum_\alpha \omega_\alpha \bar{\omega}_\alpha$, where $\alpha, \beta, \cdots$ run from 1 through n. We denote by $\{\omega_{\alpha\beta}\}$ the unitary connection forms with respect to $\{\omega_\alpha\}$. Then we have

$$
\begin{aligned}
&d\omega_\alpha = \sum \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \bar{\omega}_{\beta\alpha} = 0, \\
&d\omega_{\alpha\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \qquad (5.1) \\
&\Omega_{\alpha\beta} = -\rho(\omega_\alpha \wedge \bar{\omega}_\beta + \delta_{\alpha\beta} \sum \omega_\gamma \wedge \bar{\omega}_\gamma).
\end{aligned}
$$

We set

$$
\omega_\alpha = \theta_{2\alpha-1} + i\theta_{2\alpha},
$$

$$
\omega_{\alpha\beta} = \theta_{2\alpha-1,2\beta-1} + i\theta_{2\alpha,2\beta-1},
$$

where $i = \sqrt{-1}$. Then $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$ is a canonical 1-form of the underlying Riemannian structure of $\mathbf{X}$ and $\{\theta_{2\alpha-1,2\beta-1}, \theta_{2\alpha,2\beta-1}\}$ is the Riemannian connection form with respect to $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$. Let $\{e_{2\alpha-1}, e_{2\alpha}\}$ be the dual frame of $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$, which is an orthonormal frame with $Je_{2\alpha-1} = e_{2\alpha}$, where J is the complex structure of $\mathbf{X}$. Such a frame is called a *J-canonical frame.*

Let U be a neighbourhood of a point p of $\mathbf{X}$. We choose and fix a local orthonormal system $\{\tilde{e}_1, \tilde{e}_2\}$ of vector fields on U which may not satisfy $J\tilde{e}_1 = \tilde{e}_2$. Generalizing the notion of the Kaehler function of an immersion x introduced in Introduction, we define the function $\cos(\alpha)$ by

$$\cos(\alpha) = < J\tilde{e}_1, \tilde{e}_2 >,$$

where $< \ , \ >$ denotes the Kaehler metric of $\mathbf{X}$. We denote by $O^{(1)}_{J,p}$ the subspace of the tangent space $T_p\mathbf{X}$ spanned by $\tilde{e}_1, \tilde{e}_2, J\tilde{e}_1$ and $J\tilde{e}_2$. If $\cos^2(\alpha) \neq 1$ on U, then the dimension of $O^{(1)}_{J,p}$ is equal to 4 for each $p \in U$. Let $N^{(1)}_{J,p}$ be the orthogonal complement of $O^{(1)}_{J,p}$ in $T_p\mathbf{X}$ so that $T_p\mathbf{X} = O^{(1)}_{J,p} \oplus N^{(1)}_{J,p}$. Since $O^{(1)}_{J,p}$ and $N^{(1)}_{J,p}$ are J-invariant subspaces of $T_p\mathbf{X}$, we can define vectors $\tilde{e}_3, \tilde{e}_4, e_1, e_2, e_3$ and e_4 as follows:

$$\tilde{e}_3 = -\cot(\alpha)\tilde{e}_1 - \operatorname{cosec}(\alpha)J\tilde{e}_2,$$

$$\tilde{e}_4 = \operatorname{cosec}(\alpha)J\tilde{e}_1 - \cot(\alpha)\tilde{e}_2,$$

$$\begin{aligned} e_1 &= \cos(\frac{\alpha}{2})\tilde{e}_1 + \sin(\frac{\alpha}{2})\tilde{e}_3, \\ e_2 &= \cos(\frac{\alpha}{2})\tilde{e}_2 + \sin(\frac{\alpha}{2})\tilde{e}_4, \\ e_3 &= \sin(\frac{\alpha}{2})\tilde{e}_1 - \cos(\frac{\alpha}{2})\tilde{e}_3, \\ e_4 &= -\sin(\frac{\alpha}{2})\tilde{e}_2 + \cos(\frac{\alpha}{2})\tilde{e}_4. \end{aligned} \tag{5.2}$$

Then $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is an orthonormal basis of $O^{(1)}_{J,p}$ and $\{e_1, e_2, e_3, e_4\}$ is a J-canonical basis of $O^{(1)}_{J,p}$ for $p \in U$. This shows that *starting from any orthonormal system $\{\tilde{e}_1, \tilde{e}_2\}$ of vectors satisfying $< J\tilde{e}_1, \tilde{e}_2 > \neq \pm 1$ on U, we can construct a 4-dimensional subspace $O^{(1)}_{J,p}$ of $T_p\mathbf{X}$ generated by $\{\tilde{e}_1, \tilde{e}_2, J\tilde{e}_1, J\tilde{e}_2\}$ which has a J-canonical basis $\{e_1, e_2, e_3, e_4\}$.* Let $\{\tilde{e}_A\}$ be a local orthonormal frame on $\mathbf{X}$ which extends $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, where A runs from 1 through 2n. Let $\{\tilde{\theta}_A\}$ denote its dual frame. Then $\{e_1, e_2, e_3, e_4; \tilde{e}_\lambda, \lambda \geq 5\}$ is a local orthonormal frame such that $\{e_1, e_2, e_3, e_4\}$ is J-canonical. Putting

$$\omega_\alpha = \theta_{2\alpha-1} + i\theta_{2\alpha},$$

we have, by (5.2),

$$\begin{aligned}
&\tilde{\theta}_1 + i\tilde{\theta}_2 = \cos(\frac{\alpha}{2})\omega_1 + \sin(\frac{\alpha}{2})\bar{\omega}_2, \\
(5.3) \qquad &\tilde{\theta}_3 + i\tilde{\theta}_4 = \sin(\frac{\alpha}{2})\omega_1 - \cos(\frac{\alpha}{2})\bar{\omega}_2, \\
&\tilde{\theta}_{2\lambda-1} + i\tilde{\theta}_{2\lambda} = \omega_\lambda \ (\lambda \geq 3).
\end{aligned}$$

We set

$$\cos(\beta) = < J\tilde{e}_5, \tilde{e}_6 > .$$

If $\cos^2(\beta) \neq 1$ on an open subset U' of U, then in the same way as above the subspace $N^{(1)}_{J,p}$ has a splitting with respect to $\{\tilde{e}_5, \tilde{e}_6\}$ such that $N^{(1)}_{J,p} = O^{(2)}_{J,p} \oplus N^{(2)}_{J,p}, p \in U'$, $O^{(2)}_{J,p}$ is a J-invariant 4-dimensional subspace of $N^{(1)}_{J,p}$ spanned by $\{\tilde{e}_5, \tilde{e}_6, J\tilde{e}_5, J\tilde{e}_6\}$ and $N^{(2)}_{J,p}$ is its orthogonal complement in $N^{(1)}_{J,p}$. Thus we have an orthonormal basis $\{\tilde{e}_5, \tilde{e}_6, J\tilde{e}_7, J\tilde{e}_8\}$ and a J-canonical basis $\{e_5, e_6, e_7, e_8\}$ of $O^{(2)}_{J,p}$ over U'. Let $\{e_{2\lambda-1}, e_{2\lambda}\}(\lambda \geq 5)$ be a J-canonical basis of $\mathrm{N}^{(2)}_J$ over U and put $\tilde{e}_{2\lambda-1} = e_{2\lambda-1}$ and $\tilde{e}_{2\lambda} = e_{2\lambda}$ for $\lambda \geq 5$. Let $\{\tilde{\theta}_{2\alpha-1}, \tilde{\theta}_{2\alpha}\}$ and $\{\theta_{2\alpha-1}, \theta_{2\alpha}\}$ be dual coframes over U of $\{\tilde{e}_{2\alpha-1}, \tilde{e}_{2\alpha}\}$ and $\{e_{2\alpha-1}, e_{2\alpha}\}$, respectively. By (5.2), for $\omega_\alpha = \theta_{2\alpha-1} + i\theta_{2\alpha}$ we have the following relations :

$$\begin{aligned}
&\tilde{\theta}_1 + i\tilde{\theta}_2 = \cos(\frac{\alpha}{2})\omega_1 + \sin(\frac{\alpha}{2})\bar{\omega}_2, \\
&\tilde{\theta}_3 + i\tilde{\theta}_4 = \sin(\frac{\alpha}{2})\omega_1 - \cos(\frac{\alpha}{2})\bar{\omega}_2, \\
(5.4) \qquad &\tilde{\theta}_5 + i\tilde{\theta}_6 = \cos(\frac{\beta}{2})\omega_3 + \sin(\frac{\beta}{2})\bar{\omega}_4,
\end{aligned}$$

$$\tilde{\theta}_7 + i\tilde{\theta}_8 = \sin(\frac{\beta}{2})\omega_3 - \cos(\frac{\beta}{2})\bar{\omega}_4,$$

$$\tilde{\theta}_{2\lambda-1} + i\tilde{\theta}_{2\lambda} = \omega_\lambda \ (\lambda \geq 5).$$

Let $\{\tilde{\theta}_{2\alpha-1,2\beta-1}, \tilde{\theta}_{2\alpha-1,2\alpha}, \tilde{\theta}_{2\alpha,2\beta}\}$ be the Riemannian connection form with respect to the orthonormal coframe $\{\tilde{\theta}_{2\alpha-1}, \tilde{\theta}_{2\alpha}\}$. By taking the exterior derivative of the first equation in (5.4) and using (5.1) and (5.4), we get

$$\begin{aligned}
&\tilde{\theta}_{12} = i\{\cos^2(\frac{\alpha}{2})\omega_{11} - \sin^2(\frac{\alpha}{2})\omega_{22}\},\\
&\tilde{\theta}_{13} + i\tilde{\theta}_{23} = -\{\omega_{12} + \frac{1}{2}(d\alpha - \sin(\alpha)(\omega_{11} + \omega_{22}))\},\\
&\tilde{\theta}_{14} + i\tilde{\theta}_{24} = i\{\omega_{12} - \frac{1}{2}(d\alpha - \sin(\alpha)(\omega_{11} + \omega_{22}))\},\\
&\tilde{\theta}_{15} + i\tilde{\theta}_{25} = \{\cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\omega_{13} + \cos(\frac{\alpha}{2})\sin(\frac{\beta}{2})\omega_{14}\\
&\qquad + \sin(\frac{\alpha}{2})\cos(\frac{\beta}{2})\bar{\omega}_{23} + \sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})\bar{\omega}_{24}\},\\
(5.5)\qquad &\tilde{\theta}_{16} + i\tilde{\theta}_{26} = i\{\cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\omega_{13} - \cos(\frac{\alpha}{2})\sin(\frac{\beta}{2})\omega_{14}\\
&\qquad - \sin(\frac{\alpha}{2})\cos(\frac{\beta}{2})\bar{\omega}_{23} + \sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})\bar{\omega}_{24}\},\\
&\tilde{\theta}_{17} + i\tilde{\theta}_{27} = \{\cos(\frac{\alpha}{2})\sin(\frac{\beta}{2})\omega_{13} - \cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\omega_{14}\\
&\qquad + \sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})\bar{\omega}_{23} - \sin(\frac{\alpha}{2})\cos(\frac{\beta}{2})\bar{\omega}_{24}\},\\
&\tilde{\theta}_{18} + i\tilde{\theta}_{28} = i\{\cos(\frac{\alpha}{2})\sin(\frac{\beta}{2})\omega_{13} + \cos(\frac{\alpha}{2})\cos(\frac{\beta}{2})\omega_{14}\\
&\qquad - \sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})\bar{\omega}_{23} - \sin(\frac{\alpha}{2})\cos(\frac{\beta}{2})\bar{\omega}_{24}\},\\
&\tilde{\theta}_{1,2\lambda-1} + i\tilde{\theta}_{2,2\lambda-1} = \cos(\frac{\alpha}{2})\omega_{1\lambda} + \sin(\frac{\alpha}{2})\bar{\omega}_{2\lambda},\\
&\tilde{\theta}_{1,2\lambda} + i\tilde{\theta}_{2,2\lambda} = i\{\cos(\frac{\alpha}{2})\omega_{1\lambda} - \sin(\frac{\alpha}{2})\bar{\omega}_{2\lambda}\},\ (\lambda \geq 5).
\end{aligned}$$

By taking the exterior derivatives of (5.4), we also get identities related to $\tilde{\theta}_{\lambda\mu}$ and $\omega_{\lambda\mu}$, which we omit to list.

6. Minimal surfaces in a Kaehler manifold

In [18] Chern and Wolfson have obtained several formulas for the Gauss and Codazzi equations of an isometric minimal immersion of a real 2-dimensional Riemannian manifold into an n-dimensional Kaehler manifold using the notion of Kaehler function. In this section we extend their methods more generally to the case of an arbitrary isometric immersion and using it, we calculate the components of the second fundamental form in terms of several scaler invariants defined locally on an immersed surface, which enable us to get further information about an isometric minimal immersion.

Let M be an oriented 2-dimensional Riemannian manifold and $x : \mathrm{M} \longrightarrow \mathrm{X}$ an isometric immersion of M into a Kaehler manifold X of constant holomorphic sectional curvature 4ρ. Let $\{\tilde{e}_1, \tilde{e}_2\}$ be a local orthonormal frame on M. Hereafter, we identify $\tilde{e}_1$ and $\tilde{e}_2$ with $x_*(\tilde{e}_1)$ and $x_*(\tilde{e}_2)$, respectively. By definition, $\cos(\alpha) = < J\tilde{e}_1, \tilde{e}_2 >$ is the *Kaehler function* of x (and α is called the *Kaehler angle* of x) (cf. [18]). The immersion is said to be *totally real* if $\cos(\alpha) = 0$ on M, and to be *complex* if $\cos^2(\alpha) = 1$ on M, respectively. We assume that x is not a complex immersion at a point $p \in \mathrm{M}$. In the open subset where $\cos^2(\alpha) \neq 1$, we extend $\{\tilde{e}_1, \tilde{e}_2\}$ to a neighbourhood of $x(p)$ in X and, using the results of Section 5, we get the canonical 1-forms $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4\}$ defined on the neighbourhood. Let $\{\tilde{\theta}_A\}$ be a local frame on X which contains the $\{\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4\}$. We denote the restriction of $\{\tilde{\theta}_A\}$ to M by the same letters. Then we have $\tilde{\theta}_t = 0$ $(3 \leq t \leq 2n)$ on M. Putting $\phi = \tilde{\theta}_1 + i\tilde{\theta}_2$, the metric of M is written as $ds^2 = \phi\bar{\phi}$. By taking the exterior derivative of (5.3) restricted to M, we get

$$\frac{1}{2}\{d\alpha + \sin(\alpha)(\omega_{11} + \omega_{22})\} = a\phi + b\bar{\phi},$$

$$\omega_{12} = b\phi + c\bar{\phi}, \tag{6.1}$$

$$\cos(\frac{\alpha}{2})\omega_{\lambda 1} = a_\lambda \phi + b_\lambda \bar{\phi},$$

$$\sin(\frac{\alpha}{2})\omega_{\lambda 2} = b_\lambda \phi + c_\lambda \bar{\phi}, \; 3 \le \lambda \le n,$$

where $a, b, c, a_\lambda, b_\lambda$ and c_λ are complex-valued smooth functions defined locally on M and depend only on the choice of $\{\tilde{e}_1, \tilde{e}_2\}$. Let h_{tij}'s be the components of the second fundamental form of x so that $\tilde{\theta}_{it} = \sum_j h_{tij}\tilde{\theta}_j$. By making use of (5.5) and (6.1), all h_{tij}'s can be expressed in terms of $a, b, c, a_\lambda, b_\lambda$ and c_λ. Indeed, we have

$$h_{311} = -\frac{1}{2}\{a + \bar{a} + 2(b + \bar{b}) + c + \bar{c}\},$$

$$h_{312} = \frac{i}{2}(-a + \bar{a} + c - \bar{c}),$$

$$h_{322} = -\frac{1}{2}\{-a - \bar{a} + 2(b + \bar{b}) - c - \bar{c}\},$$

$$h_{411} = \frac{i}{2}\{a - \bar{a} + 2(b - \bar{b}) + c - \bar{c}\},$$

$$h_{412} = \frac{1}{2}(-a - \bar{a} + c + \bar{c}),$$

$$h_{422} = \frac{i}{2}\{-a + \bar{a} + 2(b - \bar{b}) - c + \bar{c}\}, \tag{6.2}$$

$$h_{2\lambda-1,11} = -\frac{1}{2}\{a_\lambda + \bar{a}_\lambda + 2(b_\lambda + \bar{b}_\lambda) + c_\lambda + \bar{c}_\lambda\},$$

$$h_{2\lambda-1,12} = \frac{i}{2}(-a_\lambda + \bar{a}_\lambda + c_\lambda - \bar{c}_\lambda),$$

$$h_{2\lambda-1,22} = -\frac{1}{2}\{-a_\lambda - \bar{a}_\lambda + 2(b_\lambda + \bar{b}_\lambda) - c_\lambda - \bar{c}_\lambda\},$$

$$h_{2\lambda,11} = \frac{i}{2}\{a_\lambda - \bar{a}_\lambda + 2(b_\lambda - \bar{b}_\lambda) + c_\lambda - \bar{c}_\lambda\},$$

$$h_{2\lambda,12} = \frac{1}{2}(-a_\lambda - \bar{a}_\lambda + c_\lambda + \bar{c}_\lambda),$$

$$h_{2\lambda,22} = \frac{i}{2}\{-a_\lambda + \bar{a}_\lambda + 2(b_\lambda - \bar{b}_\lambda) - c_\lambda + \bar{c}_\lambda\}.$$

By (6.2), the mean curvature vector H of this immersion is written as

$$\mathrm{H} = 2Re\{-\bar{b}(\tilde{e}_3 + i\tilde{e}_4) - \sum_\lambda \bar{b}_\lambda(\tilde{e}_{2\lambda-1} + i\tilde{e}_{2\lambda})\}. \tag{6.3}$$

The immersion x is said to be *minimal* if $h_{t11} + h_{t22} = 0$ on M for any t, or equivalently, if $b = b_\lambda = 0$ on M for any λ. x is said to be *superminimal* if it is minimal and $c = 0$ on M (cf. [18], [24]). Note that a complex immersion is always minimal and $|c|^2$ is a scalar invariant of x.

We are now going to derive detailed information about an isometric minimal immersion, which will be used in the following sections. From now on we assume that x is minimal. Let K be the Gaussian curvature of M defined by

$$d\tilde{\theta}_{12} = -\frac{i}{2}\mathrm{K}\phi \wedge \bar{\phi}.$$

By virtue of the first equation of (5.4) and (6.1), the Gauss equation of x takes the form (cf. Prop.1 of [24])

$$\mathrm{K} = (1 + 3\cos^2(\alpha))\rho - 2(|a|^2 + |c|^2 + \sum_\lambda |a_\lambda|^2 + \sum_\lambda |c_\lambda|^2). \tag{6.4}$$

By taking the exterior derivative of (6.1) and using the structure equations (5.1), we get, for some locally defined functions $a_i, c_i, a_{\lambda,i}$ and $c_{\lambda,i}$ $(i = 1, 2)$,

$$\begin{aligned}
&da - ia\tilde{\theta}_{12} = a_1\phi + a_2\bar{\phi},\\
&\qquad \text{with } a_2 = |a|^2\cot(\alpha) - \sum_\lambda |a_\lambda|^2 \tan(\frac{\alpha}{2})\\
&\qquad\qquad + \sum_\lambda |c_\lambda|^2 \cot(\frac{\alpha}{2}) + \frac{3}{4}\rho\sin(2\alpha),\\
&dc + 3ic\tilde{\theta}_{12} = c_1\phi + c_2\bar{\phi}, \ \text{ with } c_1 = -ac\cot(\alpha),\\
&da_\lambda - 2ia_\lambda\tilde{\theta}_{12} - \sum_\mu a_\mu\omega_{\lambda\mu} = a_{\lambda,1}\phi + a_{\lambda,2}\bar{\phi}, \ \text{ with } a_{\lambda,2} = -\bar{c}a_\lambda\cot(\frac{\alpha}{2}),\\
&dc_\lambda + 2ic_\lambda\tilde{\theta}_{12} - \sum_\mu c_\mu\omega_{\lambda\mu} = c_{\lambda,1}\phi + c_{\lambda,2}\bar{\phi}, \ \text{ with } c_{\lambda,1} = ca_\lambda\tan(\frac{\alpha}{2}).
\end{aligned} \tag{6.5}$$

Now we consider a transformation of frames on M. We put

$$\tilde{\phi} = e^{i\tau}\phi$$

and

$$\tilde{\omega}_\lambda = \sum_\mu a_{\lambda\mu}\omega_\mu,$$

where τ is a locally defined real-valued function and $(a_{\lambda\mu})$ is a unitary matrix $(\lambda, \mu \geq 3)$. Then we have

$$\tilde{\omega}_1 = e^{i\tau}\omega_1, \quad \tilde{\omega}_2 = e^{-i\tau}\omega_2,$$

and hence the first equation of (5.1) implies

$$\tilde{\omega}_{11} = id\tau + \omega_{11},$$

$$\tilde{\omega}_{22} = -id\tau + \omega_{22},$$

$$\tilde{\omega}_{12} = e^{2i\tau}\omega_{12},$$

$$\sum_\mu \tilde{\omega}_{1\mu}a_{\mu\nu} = e^{i\tau}\omega_{1\nu},$$

$$\sum_\mu \tilde{\omega}_{2\mu}a_{\mu\nu} = e^{-i\tau}\omega_{2\nu}.$$

By (6.1) we have

$$\tilde{a} = e^{-i\tau}a, \ \tilde{c} = e^{3i\tau}c, \ \tilde{a}_\lambda = e^{-2i\tau}a_{\lambda\mu}a_\mu \ \text{ and } \ \tilde{c}_\lambda = e^{2i\tau}a_{\lambda\mu}c_\mu. \tag{6.6}$$

Thus $|a|^2, |c|^2, \sum |a_\lambda|^2$ and $\sum |c_\lambda|^2$ are scalar invariants of x. We are going to compute the Laplacian of these functions.

Lemma 6.1. *Let* M *be an oriented* 2*-dimensional Riemannian manifold and* $x : \mathrm{M} \longrightarrow \mathrm{X}$ *an isometric minimal immersion of* M *into a Kaehler manifold* X *of constant holomorphic sectional curvature* 4ρ *with the Kaehler function* $\cos(\alpha)$. *Let* Δ *be the Laplacian for the metric of* M. *Then we have*

$$\Delta\alpha = 4|a|^2\cot(\alpha) - 4\sum_\lambda |a_\lambda|^2\tan(\frac{\alpha}{2}) + 4\sum_\lambda |c_\lambda|^2\cot(\frac{\alpha}{2}) + 3\rho\sin(2\alpha),$$

$$\begin{aligned}\Delta\log(|c|^2) = {} & 6\mathrm{K} + 8|a|^2 + 4\sum_\lambda |a_\lambda|^2\cos(\alpha)\sec^2(\frac{\alpha}{2}) - 4\sum_\lambda |c_\lambda|^2\cos(\alpha)cosec^2(\frac{\alpha}{2}) \\ & - 12\rho\cos^2(\alpha).\end{aligned}$$

Proof. By adding the first equation of (6.1) and its conjugate, we get $d\alpha = a\phi + \bar{a}\bar{\phi}$ and $d^c\alpha = i(\bar{a}\bar{\phi} - a\phi)$. Because of $dd^c\alpha = (i/2)(\Delta\alpha)\phi\wedge\bar{\phi}$, we get the formula for $\Delta\alpha$ by the first equation of (6.5). By the second equation of (6.5), the formula for $\Delta\log(|c|^2)$ is obtained. q.e.d.

Remark. The first formula in Lemma 6.1 was proved also by Chern and Wolfson [18, p.72]. Using this we get formulas for $\Delta\log(\sin(\alpha/2))$ and $\Delta\log(\cos(\alpha/2))$, which coincide with the formulas (2.1) and (2.2) in [20], if $n = 2$.

By making use of Lemma 6.1, we have $\Delta\log(|c|^2\sin^2(\alpha)) = 6\mathrm{K}$, which coincides with (2.2) in [24]. Hence in the same way as used in Theorem 3 of [24], we get the following.

Proposition 6.2. *Let* $\mathbf{M}$ *be a complete connected oriented* 2*-dimensional Riemannian manifold and* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *an isometric minimal immersion of* $\mathbf{M}$ *into a Kaehler manifold* $\mathbf{X}$ *of constant holomorphic sectional curvature* 4ρ*, which is not complex. If the Gaussian curvature* K *of* $\mathbf{M}$ *satisfies* $\mathrm{K} \geq 0$*, then either* $c = 0$ *or* $\mathrm{K} = 0$ *on* $\mathbf{M}$.

Note that Proposition 6.2 is an extension of Theorem 6.1 in [20] as well as Theorem 3 in [24].

We assume that $\mathbf{M}$ satisfies the same assumption as in Proposition 6.2 and $\mathrm{K} > 0$ on $\mathbf{M}$, which implies $c = 0$ by Proposition 6.2. Let

$$H^{(1)}(t) = h_{t11} + ih_{t12}$$

with $t = 3, 4, \cdots, 2\mathrm{n}$, and we put $H^{(1)} = \sum_t (H^{(1)}(t))^2$. Then we get

$$H^{(1)} = 4\sum_\lambda \bar{a}_\lambda c_\lambda$$

by (6.2). By (6.6), $|H^{(1)}|^2$ is a globally defined smooth function on $\mathbf{M}$. The geometric meaning of this function may be seen as follows. We put

$$V_1^{(1)} = \sum_t h_{t11}\tilde{e}_t \quad \text{and} \quad V_2^{(1)} = \sum_t h_{t12}\tilde{e}_t.$$

By (6.2), we have

$$(6.7) \quad \begin{aligned} V_1^{(1)} &= -\frac{1}{2}\sum_\lambda(a_\lambda+\bar{a}_\lambda+c_\lambda+\bar{c}_\lambda)\tilde{e}_{2\lambda-1} + \frac{i}{2}\sum_\lambda(a_\lambda-\bar{a}_\lambda+c_\lambda-\bar{c}_\lambda)\tilde{e}_{2\lambda}, \\ V_2^{(1)} &= -\frac{i}{2}\sum_\lambda(a_\lambda-\bar{a}_\lambda-c_\lambda+\bar{c}_\lambda)\tilde{e}_{2\lambda-1} - \frac{1}{2}\sum_\lambda(a_\lambda+\bar{a}_\lambda-c_\lambda-\bar{c}_\lambda)\tilde{e}_{2\lambda}. \end{aligned}$$

$V_1^{(1)}$ and $V_2^{(1)}$ are independent of the choice of the normal frame field $\{\tilde{e}_t\}(t \geq 3)$. The subspace $O_{J,p}^{(2)}$ spanned by $\{V_1^{(1)}, V_2^{(1)}, JV_1^{(1)}, JV_2^{(1)}\}$ is called the *J-invariant first normal space* of x. The geometric meaning of $|H^{(1)}|^2$ follows from the identity

$$|H^{(1)}|^2 = (|V_1^{(1)}|^2 - |V_2^{(1)}|^2)^2 + 4 < V_1^{(1)}, V_2^{(1)} >^2 .$$

It follows from (6.5)

$$dH^{(1)} + 4iH^{(1)}\tilde{\theta}_{12} = \bar{H}_2^{(1)}\bar{\phi},$$

where we put $H_2^{(1)} = 4\sum(\bar{a}_\lambda c_{\lambda,2} + \bar{a}_{\lambda,1}c_\lambda)$. Hence

$$\Delta|H^{(1)}|^2 = 2(4\mathrm{K}|H^{(1)}|^2 + 2|H_2^{(1)}|^2).$$

On the other hand, we have $H^{(1)} \leq 4(\sum|a_\lambda|^2 + \sum|c_\lambda|^2)^2$ by Schwarz's inequality. From these and the Gauss equation (6.4), if $\mathrm{K} > 0$, $|H^{(1)}|^2$ is a subharmonic function on M bounded above, hence is constant ($= 0$).

We define a subset of M by

$$\Omega_{(2)} = \{p \in \mathrm{M} : V_1^{(1)}(p) = 0 \text{ or } V_2^{(1)}(p) = 0\}.$$

For the set $T_p^1(\mathrm{M})$ of unit tangent vectors of $T_p(\mathrm{M})$, we define a subset of $N_p(\mathrm{M})$ by

$$A(T_p^1(\mathrm{M})) = \{\sum_{t,i,j} h_{tij}X_iX_j\tilde{e}_t : \ \sum_i X_i\tilde{e}_i \in T_p^1(\mathrm{M})\},$$

which is called the *ellipse of curvature in the first normal space* ([20]). Summarizing these computations, we have the following:

Proposition 6.3. *Under the same assumption as in Proposition* 6.2, *if the Gaussian curvature* K *satisfies* $\mathrm{K} > 0$ *on* $\mathbf{M}$ *and* $\Omega_{(2)} = \emptyset$, *then the ellipse of curvature in the first normal space is a circle.*

Now we shall describe a family of full isometric minimal immersions of two-sphere $\mathbf{S}^2(\mathrm{K})$ of constant Gaussian curvature K into $\mathbf{P}^n(\mathbf{C})$, where $\mathbf{P}^n(\mathbf{C})$ is the n-dimensional complex space with the Fubini-Study metric of constant holomorphic sectional curvature 4ρ. It is well-known that there are two classes of such immersions of $\mathbf{S}^2(\mathrm{K})$ into $\mathbf{P}^n(\mathbf{C})$ (cf.[2]). One is the class of holomorphic isometric imbeddings of $\mathbf{P}^1(\mathbf{C})$ into $\mathbf{P}^n(\mathbf{C})$:

$$\xi_n : \mathbf{P}^1(\mathbf{C}) \longrightarrow \mathbf{P}^n(\mathbf{C}) \tag{6.8}$$

defined by

$$\xi_n(z_0, z_1) = \left(\sqrt{\frac{n!}{l!(n-l)!}} z_0^l z_1^{n-l} \right)_{l=0,\cdots,n},$$

where (z_0, z_1) is the homogeneous coordinate system of $\mathbf{P}^1(\mathbf{C})$. The Gaussian curvature of this immersion is 4/n for the induced metric. By using the Hopf fibration $\pi : \mathbf{C}^2 \longrightarrow \mathbf{S}^2$, we may identify $\mathbf{P}^1(\mathbf{C})$ and $\mathbf{S}^2(4/n)$.

The other is the class of totally real isometric minimal immersions μ_k obtained by composing a generalized Veronese surface (cf. Chap I. Theorem 1.3)

$$\mathbf{S}^2(\mathrm{K}(k)) \longrightarrow \mathbf{S}^{2k}$$

with the natural covering $\mathbf{S}^{2k} \longrightarrow \mathbf{P}^{2k}(\mathbf{R})$ and a totally real, totally geodesic imbedding $\mathbf{P}^{2k}(\mathbf{R}) \longrightarrow \mathbf{P}^{2k}(\mathbf{C})$:

$$\mu_k : \mathbf{S}^2(\mathrm{K}(k)) \longrightarrow \mathbf{P}^{2k}(\mathbf{C}). \tag{6.9}$$

These immersions do not exhaust all isometric minimal immersions of the two-sphere with constant Gaussian curvature into $\mathbf{P}^n(\mathbf{C})$. In fact, by using the theory of harmonic mappings of $\mathbf{S}^2$ into $\mathbf{P}^n(\mathbf{C})$, Bando–Ohnita [2] have constructed examples of minimal two-spheres with constant Gaussian curvature in $\mathbf{P}^n(\mathbf{C})$ which are neither holomorphic nor totally real. We remark that Bolton–Jensen–Rigoli–Woodward [5] have also found the same examples later. We put

$$\mathrm{K}_{n,k} = \frac{4\rho}{n + 2k(n-k)}$$

for a positive integer k.

Theorem 6.4 ([2] and [5]). *For any non-negative integers n and k with $0 \leq k \leq n$, there exists a full isometric minimal immersion*

$$\varphi_{n,k} : \mathbf{S}^2(\mathrm{K}_{n,k}) \longrightarrow \mathbf{P}^n(\mathbf{C}),$$

and $\varphi_{n,k}$ satisfies the following:

(1) *If $k = 0$ or $k = n$, then $\varphi_{n,k}$ is holomorphic (with respect to a suitable fixed complex structure of $\mathbf{S}^2(\mathrm{K}_{n,k})$) and $\varphi_{n,k}$ is congruent to ξ_n.*

(2) *If n is even and $k = n/2$, then $\varphi_{n,k}$ is totally real and $\varphi_{n,k}$ is congruent to μ_k.*

(3) *In other cases, $\varphi_{n,k}$ is neither holomorphic nor totally real.*

Moreover they proved the following rigidity theorm.

Theorem 6.5 ([2] and [5]). *Let $x : \mathbf{S}^2(\mathrm{K}) \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a full isometric minimal immersion of the two-sphere $\mathbf{S}^2$ of constant Gaussian curvature K into $\mathbf{P}^n(\mathbf{C})$. Then there exists an integer k with $0 \leq k \leq n$ such that K is equal to $\mathrm{K}_{n,k}$, and x is congruent to $\varphi_{n,k}$.*

It should be remarked that the Kaehler function $\cos(\alpha_{n,k})$ of $\varphi_{n,k}$ is given by

$$\cos(\alpha_{n,k}) = \frac{n-2k}{n+2k(n-k)}\,, \tag{6.10}$$

for which the following formula holds.

$$(6.11) \qquad \mathrm{K}_{n,k} = \frac{2(1-(2k+1)\cos(\alpha_{n,k}))\rho}{k(k+1)} \ .$$

These immersions $\varphi_{n,k}$ are also called *generalized Veronese immersions* in $\mathbf{P}^n(\mathbf{C})$ and $\varphi_{n,k}(\mathrm{S}^2(\mathrm{K}_{n,k}))$ are called *generalized Veronese surfaces* in $\mathbf{P}^n(\mathbf{C})$ in this thesis.

7. Minimal surfaces with constant Kaehler function

Each generalized Veronese surface in $\mathbf{P}^n(\mathbf{C})$ has constant Gaussian curvature and constant Kaehler function satisfying (6.11). Ohnita [31] has characterized minimal surfaces in $\mathbf{P}^n(\mathbf{C})$ whose Gaussian curvature and Kaehler function are both constant and proved the following:

Theorem (Ohnita [31]). *Let* $\mathbf{M}$ *be a complete connected oriented* 2*-dimensional Riemannian manifold and* $x : \mathbf{M} \longrightarrow \mathbf{P}^n(\mathbf{C})$ *a full isometric minimal immersion of* $\mathbf{M}$ *into* $\mathbf{P}^n(\mathbf{C})$. *Assume that the Gaussian curvature* K *of* $\mathbf{M}$ *and the Kaehler function* $\cos(\alpha)$ *of* x *are both constant on* $\mathbf{M}$. *Then the following holds*:

(1) *If* $\mathrm{K} > 0$, *then there exists some* k *with* $0 \leq k \leq n$ *such that* $\mathrm{K} = \mathrm{K}_{n,k}$, $\cos(\alpha) = \cos(\alpha_{n,k})$ *and* $\varphi(\mathbf{M})$ *is an open submanifold of* $\varphi_{n,k}(\mathrm{S}^2(\mathrm{K}))$.

(2) *If* $\mathrm{K} = 0$, *then* $\cos(\alpha) = 0$, *that is,* φ *is totally real.* (*Such* φ*'s have already been classified by Kenmotsu* [24].)

(3) *The case of* $\mathrm{K} < 0$ *is impossible.*

Later Bolton et al. [5] conjectured that if the Kaehler function is constant, then the Gaussian curvature is also constant, provided that the immersion is neither holomorphic, anti-holomorphic nor totally real. They proved this conjecture affirmatively for n ≤ 4. We shall study this conjecture by examining properties of the pinched Gaussian curvature. This problem is a complex-version of the Simon's conjecture discussed in Chapter I.

Let $x : \mathrm{M} \longrightarrow \mathbf{X}$ be an isometric minimal immersion of a 2-dimensional Riemannian manifold $\mathbf{M}$ into a Kaehler manifold $\mathbf{X}$ of constant holomorphic sectional curvature 4ρ with constant Kaehler function $\cos(\alpha)$. Suppose that the immersion x is not complex and the Gaussian curvature K satisfies $\mathrm{K} > 0$ on $\mathbf{M}$. Since $\cos(\alpha) =$ constant, we get $a = 0$. Then by Lemma 6.1 and Proposition 6.2 we have

$$4\tan(\frac{\alpha}{2})\sum|a_\lambda|^2 - 4\cot(\frac{\alpha}{2})\sum|c_\lambda|^2 - 3\rho\sin(2\alpha) = 0,$$

$$c = 0.$$

Hence the Gauss equation (6.4) is expressed as

$$\sum|a_\lambda|^2 + \sum|c_\lambda|^2 = \frac{1}{2}(1 + 3\cos^2(\alpha))\rho - \frac{1}{2}\mathrm{K}.$$

These equations imply

$$\sum|a_\lambda|^2 = \frac{1}{2}\cos^2(\frac{\alpha}{2})(\rho + 3\rho\cos(\alpha) - \mathrm{K}),$$

(7.1)

$$\sum|c_\lambda|^2 = \frac{1}{2}\sin^2(\frac{\alpha}{2})(\rho - 3\rho\cos(\alpha) - \mathrm{K}).$$

If $\mathrm{K} \geq (1 - 3\cos(\alpha))\rho > 0$, we then have $\mathrm{K} = (1 - 3\cos(\alpha))\rho$, which means that K is constant. Hence, by making use of Ohnita's theorem [31], we get:

Theorem 7.1. *Let* $\mathbf{M}$ *be a complete connected oriented 2-dimensional Riemannian manifold and* $\mathbf{X}$ *a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature* 4ρ*. Let* $x : \mathrm{M} \longrightarrow \mathbf{X}$ *be a full isometric minimal immersion with constant Kaehler function* $\cos(\alpha)$ *which is not complex. If the Gaussian curvature* K *of* $\mathbf{M}$ *is strictly positive and satisfies*

$$\mathrm{K} \geq (1 - 3\cos(\alpha))\rho,$$

then K *is constant. If*

$$\mathrm{K} \geq (1 + 3\cos(\alpha))\rho,$$

then K *is constant. When* $\mathbf{X}$ *is* $\mathbf{P}^n(\mathbf{C})$, *such an* x *is congruent to* $\varphi_{n,1}$ *or* $\varphi_{n,n-1}$, *respectively.*

The Gaussian curvature $K_{n,k}$ and the Kaehler function $\cos(\alpha_{n,k})$ of the generalized Veronese immersion $\varphi_{n,k}$ in $\mathbf{P}^n(\mathbf{C})$ satisfy the formula (6.11) except $k = 0$. For example when $k = 1$ we have the formula

$$K_{n,1} = (1 - 3\cos(\alpha_{n,1}))\rho.$$

When $k = n - 1$, $K_{n,n-1}$ and $\cos(\alpha_{n,n-1})$ satisfy both (6.11) and the relation

$$K_{n,n-1} = (1 + 3\cos(\alpha_{n,n-1}))\rho.$$

The above Theorem 7.1 gives us a characterization of the immersions $\varphi_{n,1}$ and $\varphi_{n,n-1}$ by the pinched Gaussian curvature. We are interested in studying the pinching problem of the Gaussian curvature with the next constant following $K_{n,1}$.

By virtue of Theorem 7.1, if the Kaehler function $\cos(\alpha)$ is positive constant, then there exists no isometric minimal immersion $x : \mathbf{M} \longrightarrow \mathbf{P}^2(\mathbf{C})$ except complex immersions (cf. Proposition 7.3).

By (7.1) we have

$$\sum |a_\lambda|^2 - \sum |c_\lambda|^2 = \frac{1}{2}(4\rho - K)\cos(\alpha).$$

Combining this equation and (6.7) we get

$$\Omega_{(2)} = \emptyset \quad \text{if} \quad \cos(\alpha) \neq 0 \quad \text{on } \mathbf{M},$$

which means that the normal vectors $V_1^{(1)}$ and $V_2^{(1)}$ are both not zero on $\mathbf{M}$. From now on we assume that x is not totally real, i.e., $\cos(\alpha) \neq 0$. Under the assumption of Theorem 7.1, the Gauss equation (6.4) of x is expressed as

$$K = (1 + 3\cos^2(\alpha))\rho - 2(\sum_\lambda | a_\lambda |^2 + \sum_\lambda | c_\lambda |^2).$$

Hence, K may be determined by $\sum_\lambda \mid a_\lambda \mid^2$ and $\sum_\lambda \mid c_\lambda \mid^2$, which are scalar invariants of x (cf.(6.6)).

Lemma 7.2. *Under the same assumption as in Theorem* 7.1, *we have*

$$\Delta(\sum |a_\lambda|^2) = 2(3\mathrm{K} - \rho - 5\rho\cos(\alpha))(\sum |a_\lambda|^2) + 4\sum |a_{\lambda,1}|^2,$$

$$\Delta(\sum |c_\lambda|^2) = 2(3\mathrm{K} - \rho + 5\rho\cos(\alpha))(\sum |c_\lambda|^2) + 4\sum |c_{\lambda,2}|^2.$$

Proof. We will prove only the formula for $\Delta(\sum |c_\lambda|^2)$, because the other can be shown in a similar way. By the forth formula of (6.5), we have

$$d(\sum |c_\lambda|^2) = \sum_\lambda \{(c_\lambda \bar{c}_{\lambda,2} + \bar{c}_\lambda c_{\lambda,1})\phi + (c_\lambda \bar{c}_{\lambda,1} + \bar{c}_\lambda c_{\lambda,2})\bar{\phi}\}$$

and

$$dc_{\lambda,1} + ic_{\lambda,1}\tilde{\theta}_{12} - \sum_\mu c_{\mu,1}\omega_{\lambda\mu} = \{\tan(\frac{\alpha}{2})a_\lambda c_1 + \tan(\frac{\alpha}{2})a_{\lambda,1}c + \frac{1}{2}\sec^2(\frac{\alpha}{2})aca_\lambda\}\phi$$

$$+ \{\tan(\frac{\alpha}{2})a_\lambda c_2 + \tan(\frac{\alpha}{2})a_{\lambda,2}c + \frac{1}{2}\sec^2(\frac{\alpha}{2})\bar{a}ca_\lambda\}\bar{\phi}.$$

Hence we obtain

$$dd^c(\sum |c_\lambda|^2) = 2i\{(\sum |c_\lambda|^2)\mathrm{K} + \sum(|c_{\lambda,1}|^2 + |c_{\lambda,2}|^2) + (L + \bar{L})$$

$$+ \sec^2(\frac{\alpha}{2})|\sum a_\lambda \bar{c}_\lambda|^2 - \operatorname{cosec}^2(\frac{\alpha}{2})(\sum |c_\lambda|^2)^2 + \rho\cos(\alpha)(\sum |c_\lambda|^2)\}\phi\wedge\bar{\phi},$$

where we put

$$L = \sum_\lambda \{\tan(\frac{\alpha}{2})\bar{a}_\lambda \bar{c}_2 + \tan(\frac{\alpha}{2})\bar{a}_{\lambda,2}\bar{c} + \frac{1}{2}\sec^2(\frac{\alpha}{2})a\bar{c}\bar{a}_\lambda\}c_\lambda.$$

It then follows from Lemma 6.1, Proposition 6.3 and the second formula of (7.1) that

$$c_{\lambda,1} = 0, \ \sum_\lambda a_\lambda \bar{c}_\lambda = 0 \ \text{ and } \ L = 0.$$

These formulas then imply Lemma 7.2. q.e.d.

The following Proposition 7.3 is important to obtain the information on the fullness of an isometric minimal immersion:

Proposition 7.3. *Let $x : \mathbf{M} \longrightarrow \mathbf{X}$ be a full isometric minimal immersion with constant Kaehler function $\cos(\alpha)$, which is neither complex nor totally real. If the Gaussian curvature K of $\mathbf{M}$ is strictly positive and there exists an open subset U of $\mathbf{M}$ such that on U*

$$\mathrm{K} < (1 - 3\cos(\alpha))\rho,$$

then we have $n \geq 4$.

Proof. By (7.1), we have $V_1^{(1)} \neq 0$ and $V_2^{(1)} \neq 0$ on U, and $\tilde{e}_1, \tilde{e}_2, J\tilde{e}_1, J\tilde{e}_2, V_1^{(1)}, V_2^{(1)}, JV_1^{(1)}$ and $JV_2^{(1)}$ are linearly independent on U. In consequence, we have $n \geq 4$.

q.e.d.

The next largest value of the Gaussian curvatures for the metric induced by the generalized Veronese immersions $\{\varphi_{n,k}\}$ in $\mathbf{P}^n(\mathbf{C})$ is $\mathrm{K}_{n,2}$, which satisfies

$$\mathrm{K}_{n,2} = \frac{(1 - 5\cos(\alpha_{n,2}))\rho}{3},$$

where $\cos(\alpha_{n,2})$ is the positive Kaehler function of the immersion $\varphi_{n,2}$. The following Theorem 7.4, which follows from the second formula in Lemma 7.2 and the second equation of (7.1), gives us a characterization of the immersion $\varphi_{n,2}$.

Theorem 7.4. *Let $\mathbf{X}$ be a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature 4ρ and $\mathbf{M}$ a complete connected oriented 2-dimensional Riemannian manifold. Let $x : \mathbf{M} \longrightarrow \mathbf{X}$ be a full isometric minimal immersion with constant Kaehler function , which is neither complex nor totally real. If the Gaussian curvature K of $\mathbf{M}$ is strictly positive and satisfies*

$$\mathrm{K} \geq \frac{(1 - 5\cos(\alpha))\rho}{3},$$

then K is constant, and hence $\mathrm{K} = (1 - 5\cos(\alpha))\rho/3$ or $\sum |c_\lambda|^2 = 0$. When $\mathbf{X}$ is $\mathbf{P}^n(\mathbf{C})$, such an x is congruent to $\varphi_{n,2}$ if $\mathrm{K} = (1-5\cos(\alpha))\rho/3$, or $\varphi_{n,1}$ if $\sum |c_\lambda|^2 = 0$.

Corollary 7.5. *Under the same assumption as in Theorem 7.4, if*

$$(1-3\cos(\alpha))\rho > \mathrm{K} \geq \frac{(1-5\cos(\alpha))\rho}{3},$$

then K *identically equals to* $(1-5\cos(\alpha))\rho/3$.

Remark. (1). Using the first formula in Lemma 7.2, we can prove a result analogous to Theorem 7.4. In fact, if $\mathrm{K} \geq (1+5\cos(\alpha))\rho/3$ on $\mathbf{M}$, then K is constant so that x is congruent to $\varphi_{n,n-1}$ or $\varphi_{n,n-2}$ when $\mathbf{X} = \mathbf{P}^n(\mathbf{C})$.

(2). By (6.10) and (6.11), we have

$$\mathrm{K}_{n,k} = \mathrm{K}_{n,n-k} \quad \text{and} \quad \cos(\alpha_{n,k}) = -\cos(\alpha_{n,n-k}).$$

The generalized Veronese immersion $\varphi_{n,k}$ differs from $\varphi_{n,n-k}$ by the antipodal mapping of $\mathbf{S}^2$. The formula (7.1) and Lemma 7.2 show that we may investigate $(\sum |c_\lambda|^2)$ or $(\sum |a_\lambda|^2)$ according as $\cos(\alpha) > 0$ or $\cos(\alpha) < 0$, respectively. Hence, hereafter we assume that $\cos(\alpha) > 0$.

On account of Proposition 6.3 and the assumption that x is not totally real, we have that $V_1^{(1)}$ and $V_2^{(1)}$ are perpendicular to each other and have the same length. Normalizing these vectors, we adopt them as a basis of $O_{J,p}^{(2)}$ so that

$$\tilde{e}'_5 = \frac{V_1^{(1)}}{\| V_1^{(1)} \|} \quad \text{and} \quad \tilde{e}'_6 = \frac{V_2^{(1)}}{\| V_2^{(1)} \|}.$$

We put

$$\cos(\alpha_2) = < J\tilde{e}'_5, \tilde{e}'_6 > .$$

Then we have

$$\cos(\alpha_2) = \frac{\sum |a_\lambda|^2 - \sum |c_\lambda|^2}{\sum |a_\lambda|^2 + \sum |c_\lambda|^2}.$$

If $\cos(\alpha_2) = \pm 1$ on $\mathbf{M}$, then we have $\sum |a_\lambda|^2 = 0$ or $\sum |c_\lambda|^2 = 0$. Hence this case is reduced to Theorem 7.1 by (7.1).

Now we assume $\cos(\alpha_2) \neq \pm 1$ at a point of M. Then we have $\dim(O^{(2)}_{J,p}) = 4$ in a neighbourhood U of this point. So, as in Section 5, we get the equations (5.2) and (5.4) on U. With respect to this new frame, we have

$$V_1^{(1)} = h'_{511}\tilde{e}'_5, \quad V_2^{(1)} = h'_{612}\tilde{e}'_6$$

$$h'_{611} = h'_{t11} = h'_{512} = h'_{t12} = 0 \ (t \geq 7).$$

From these equations, (5.4) and (6.1), we have

$$c_3 = \cot(\frac{\alpha_2}{2})\bar{a}_4,$$

$$c_4 = \tan(\frac{\alpha_2}{2})\bar{a}_3, \tag{7.2}$$

$$a_\lambda = c_\lambda = 0, \ (\lambda \geq 5).$$

Moreover, because of $\| V_1^{(1)} \| = \| V_2^{(1)} \|$, both c_3 and c_4 are real-valued and $c_3c_4 = 0$. We may assume $c_3 \neq 0$. Then we see

$$h'_{511} = -\sec(\frac{\alpha_2}{2})c_3 \quad \text{and} \quad h'_{612} = \sec(\frac{\alpha_2}{2})c_3.$$

Using (6.1), (6.5) and the facts mentioned above, we get

$$\sin(\frac{\alpha}{2})\omega_{32} = c_3\bar{\phi}, \quad \cos(\frac{\alpha}{2})\omega_{41} = a_4\phi,$$

$$\omega_{31} = \omega_{42} = \omega_{\lambda 1} = \omega_{\lambda 2} = 0, \ (\lambda \geq 5),$$

$$dc_3 + 2ic_3\tilde{\theta}_{12} - c_3\omega_{33} = c_{3,2}\bar{\phi}, \tag{7.3}$$

$$c_3\omega_{43} = -c_{4,2}\bar{\phi}, \quad c_3\omega_{\lambda 3} = -c_{\lambda,2}\bar{\phi}, \quad (\lambda \geq 5),$$

$$da_4 - 2ia_4\tilde{\theta}_{12} - a_4\omega_{44} = a_{4,1}\phi,$$

$$a_4\omega_{34} = -a_{3,1}\phi, \quad a_4\omega_{\lambda 4} = -a_{\lambda,1}\phi, \quad (\lambda \geq 5).$$

From now on, $\lambda, \mu, \cdots$ run from 5 through n. By taking the exterior derivatives of (7.3) and using the structure equations (5.1), we have

$$
\begin{aligned}
& dc_{4,2} + 3ic_{4,2}\tilde{\theta}_{12} - c_{4,2}\omega_{44} = c_{4,22}\bar{\phi}, \\
& dc_{\lambda,2} + 3ic_{\lambda,2}\tilde{\theta}_{12} - \sum_{\mu} c_{\mu,2}\omega_{\lambda\mu} = c_{\lambda,21}\phi + c_{\lambda,22}\bar{\phi}, \\
(7.4) \qquad & \qquad\qquad \text{with } c_{\lambda,21} = -\frac{c_{4,2}a_{\lambda,1}}{a_4}, \\
& da_{3,1} - 3ia_{3,1}\tilde{\theta}_{12} - a_{3,1}\omega_{33} = a_{3,11}\phi, \\
& da_{\lambda,1} - 3ia_{\lambda,1}\tilde{\theta}_{12} - \sum_{\mu} a_{\mu,1}\omega_{\lambda\mu} = a_{\lambda,11}\phi + a_{\lambda,12}\bar{\phi}, \\
& \qquad\qquad \text{with } a_{\lambda,12} = -\frac{a_{3,1}c_{\lambda,2}}{c_3}.
\end{aligned}
$$

By the definition of $\tilde{e}'_5$ and $\tilde{e}'_6$ we have $\tilde{\theta}_{i,2\lambda-1} = \tilde{\theta}_{i,2\lambda} = 0$ $(\lambda \geq 5)$. Taking the exterior derivatives of these forms and using the structure equations (5.1), we introduce the quantities defined by the following equations:

$$
\begin{aligned}
& h'_{511}\tilde{\theta}_{5,2\lambda-1} = h_{2\lambda-1,111}\tilde{\theta}_1 + h_{2\lambda-1,112}\tilde{\theta}_2, \\
& h'_{612}\tilde{\theta}_{6,2\lambda-1} = h_{2\lambda-1,112}\tilde{\theta}_1 - h_{2\lambda-1,111}\tilde{\theta}_2, \\
(7.5) \qquad & \\
& h'_{511}\tilde{\theta}_{5,2\lambda} = h_{2\lambda,111}\tilde{\theta}_1 + h_{2\lambda,112}\tilde{\theta}_2, \\
& h'_{612}\tilde{\theta}_{6,2\lambda} = h_{2\lambda,112}\tilde{\theta}_1 - h_{2\lambda,111}\tilde{\theta}_2, \ (\lambda \geq 5).
\end{aligned}
$$

By taking the exterior derivative of the third formula of (5.4), we get

$$
\begin{aligned}
& \tilde{\theta}_{5,2\lambda-1} + i\tilde{\theta}_{6,2\lambda-1} = \cos(\frac{\alpha_2}{2})\omega_{3\lambda} + \sin(\frac{\alpha_2}{2})\bar{\omega}_{4\lambda}, \\
(7.6) \qquad & \\
& \tilde{\theta}_{5,2\lambda} + i\tilde{\theta}_{6,2\lambda} = i\{\cos(\frac{\alpha_2}{2})\omega_{3\lambda} - \sin(\frac{\alpha_2}{2})\bar{\omega}_{4\lambda}\},
\end{aligned}
$$

where we put $\beta = \alpha_2$. By (7.3), $h_{2\lambda-1,111}$, $h_{2\lambda-1,112}$, $h_{2\lambda,111}$ and $h_{2\lambda,112}$ are expressed in terms of $a_{\lambda,1}$ and $c_{\lambda,2}$, since $h'_{511} = h'_{612} = -\sec(\alpha_2/2)c_3$. Indeed we have

$$
h_{2\lambda-1,111} = -\frac{1}{2}(a_{\lambda,1} + \bar{a}_{\lambda,1} + c_{\lambda,2} + \bar{c}_{\lambda,2}),
$$

$$h_{2\lambda-1,112} = -\frac{i}{2}(a_{\lambda,1} - \bar{a}_{\lambda,1} - c_{\lambda,2} + \bar{c}_{\lambda,2}),$$

(7.7)

$$h_{2\lambda,111} = \frac{i}{2}(a_{\lambda,1} - \bar{a}_{\lambda,1} + c_{\lambda,2} - \bar{c}_{\lambda,2}),$$

$$h_{2\lambda,112} = -\frac{1}{2}(a_{\lambda,1} + \bar{a}_{\lambda,1} - c_{\lambda,2} - \bar{c}_{\lambda,2}).$$

Using these quantities we define normal vectors $V_1^{(2)}$ and $V_2^{(2)}$ by

$$V_1^{(2)} = \sum(h_{2\lambda-1,111}\tilde{e}_{2\lambda-1} + h_{2\lambda,111}\tilde{e}_{2\lambda}),$$

$$V_2^{(2)} = \sum(h_{2\lambda-1,112}\tilde{e}_{2\lambda-1} + h_{2\lambda,112}\tilde{e}_{2\lambda}).$$

By (7.7), $V_1^{(2)}$ and $V_2^{(2)}$ are of the following form:

$$\begin{aligned} V_1^{(2)} &= -\frac{1}{2}\sum(a_{\lambda,1} + \bar{a}_{\lambda,1} + c_{\lambda,2} + \bar{c}_{\lambda,2})\tilde{e}_{2\lambda-1} \\ &\quad + \frac{i}{2}\sum(a_{\lambda,1} - \bar{a}_{\lambda,1} + c_{\lambda,2} - \bar{c}_{\lambda,2})\tilde{e}_{2\lambda}, \end{aligned}$$

(7.8)

$$\begin{aligned} V_2^{(2)} &= -\frac{i}{2}\sum(a_{\lambda,1} - \bar{a}_{\lambda,1} - c_{\lambda,2} + \bar{c}_{\lambda,2})\tilde{e}_{2\lambda-1} \\ &\quad - \frac{1}{2}\sum(a_{\lambda,1} + \bar{a}_{\lambda,1} - c_{\lambda,2} - \bar{c}_{\lambda,2})\tilde{e}_{2\lambda}. \end{aligned}$$

The following Proposition 7.6 gives information on the fullness of an isometric minimal immersion.

Proposition 7.6. *Let $x : \mathbf{M} \longrightarrow \mathbf{X}$ be a full isometric minimal immersion with constant Kaehler function* $\cos(\alpha)$*, which is neither complex nor totally real. If the Gaussian curvature* K *of* $\mathbf{M}$ *is strictly positive and if there exists an open subset* U *of* $\mathbf{M}$ *on which*

$$\mathrm{K} < \frac{(1 - 5\cos(\alpha))\rho}{3},$$

then we have $n \geq 5$.

Proof. By assumption we get $\mathrm{K} < (1 - 3\cos(\alpha))\rho$ on U. Hence by Proposition 7.3, we get $n \geq 4$ and $\sum |c_\lambda|^2 \neq 0$. Assume that $\sum |c_{\lambda,2}|^2 = 0$ on U. Then we have

$d(\sum |c_\lambda^2|) = 0$. On the other hand by Lemma 7.2 we have $\Delta(\sum |c_\lambda|^2) \neq 0$, which contradicts the constancy of $\sum |c_\lambda|^2$. Hence we have $\sum |c_{\lambda,2}|^2 \neq 0$. Using (7.7) we have $V_1^{(2)} \neq 0$ or $V_2^{(2)} \neq 0$ at a point of U. This shows that $n \geq 5$. q.e.d.

Remark. Combining Theorem 7.4 and Proposition 7.6, we can also prove that if $n \leq 4$ and the Kaehler function is constant, then the Gaussian curvature is also constant, provided the immersion is neither complex nor totally real (cf. [5]).

Now we investigate the well-definedness of scalars $|c_{4,2}|^2$ and $\sum |c_{\lambda,2}|^2$ (cf.(6.6)). Let $\{\tilde{e}'_1, \tilde{e}'_2\}$ be another local orthonormal frame on $\mathbf{M}$ such that

$$\tilde{e}'_1 = \cos(\tau)\tilde{e}_1 - \sin(\tau)\tilde{e}_2$$

$$\tilde{e}'_2 = \sin(\tau)\tilde{e}_1 + \cos(\tau)\tilde{e}_2.$$

Then we have

$$V_1'^{(1)} = \cos(2\tau)V_1^{(1)} - \sin(2\tau)V_2^{(1)}$$

$$V_2'^{(1)} = \sin(2\tau)V_1^{(1)} + \cos(2\tau)V_2^{(1)}.$$

On the other hand, by the definition of c_3, we have

$$V_1^{(1)} = -\sec(\frac{\alpha_2}{2})c_3 e_5,$$

$$V_2^{(1)} = \sec(\frac{\alpha_2}{2})c_3 e_6.$$

Under such a change of frame we have by (7.3)

$$(7.9) \qquad c'_3 = c_3, \ \ a'_4 = a_4, \ \ c'_{4,2} = e^{5i\tau}c_{4,2} \text{ and } c'_{\lambda,2} = e^{3i\tau}(\sum a_{\lambda\mu}c_{\mu,2}),$$

where we put $\omega'_\lambda = \sum a_{\lambda\mu}\omega_\mu$ for a unitary matrix $(a_{\lambda\mu})$ $(5 \leq \lambda, \mu \leq n)$. Hence we see $|c_{4,2}|^2$ *and* $\sum |c_{\lambda,2}|^2$ *are scalar invariants of the immersion* x.

In order to study properties of the third fundamental form of the immersion x, it follows from (7.7) that it suffices to study the corresponding properties of $\{c_{\lambda,2}\}$. This leads us to calculate the Laplacian of $\sum |c_{\lambda,2}|^2$.

Lemma 7.7. *Let $x : \mathrm{M} \longrightarrow \mathbf{X}$ be a full isometric minimal immersion with constant Kaehler function $\cos(\alpha)$, which is neither complex nor totally real. On an open subset U of M such that $\cos(\alpha_2) \neq \pm 1$, we have*

$$\Delta(|c_{4,2}|^2) = 6\mathrm{K}|c_{4,2}|^2 + 4|c_{4,22}|^2 + 4|c_{4,2}|^2\{\sec^2(\frac{\alpha}{2})a_4^2 - \frac{|c_{4,2}|^2}{c_3^2} - \frac{\sum |a_{\lambda,1}|^2}{a_4^2} + \rho\cos(\alpha)\},$$

$$\Delta(\sum |c_{\lambda,2}|^2) = 6\mathrm{K}(\sum |c_{\lambda,2}|^2) + 4(\sum |c_{\lambda,21}|^2 + \sum |c_{\lambda,22}|^2) + 4\rho\cos(\alpha)(\sum |c_{\lambda,2}|^2)$$

$$-\frac{4(\sum |c_{\lambda,2}|^2)^2}{c_3^2} + \frac{4|(\sum \bar{c}_{\lambda,2}a_{\lambda,1}|^2}{a_4^2} - \frac{8|c_{4,2}|^2\sum |c_{\lambda,2}|^2}{c_3^2}$$

$$-\frac{4\bar{c}_{4,22}\sum c_{\lambda,2}\bar{a}_{\lambda,1}}{a_4} - \frac{4c_{4,22}\sum \bar{c}_{\lambda,2}a_{\lambda,1}}{a_4},$$

where λ runs from 5 through n.

Proof. Under the assumptions of Lemma 7.7, we have $c_3 \neq 0$ and $a_4 \neq 0$. We here prove the formula only for $\Delta(\sum |c_{\lambda,2}|^2)$, because the other one can be shown in a similar way. By (7.3) and the second formula of (7.4), we have

$$d(\sum |c_{\lambda,2}|^2) = \sum(c_{\lambda,2}\bar{c}_{\lambda,22} + \bar{c}_{\lambda,2}c_{\lambda,21})\phi + \sum(c_{\lambda,2}\bar{c}_{\lambda,21} + \bar{c}_{\lambda,2}c_{\lambda,22})\bar{\phi},$$

$$dc_{\lambda,21} + 2ic_{\lambda,21}\tilde{\theta}_{12} - \sum c_{\mu,21}\omega_{\lambda\mu} = (-c_{4,2}\frac{a_{\lambda,11}}{a_4} + c_{4,2}\frac{a_{4,1}a_{\lambda,1}}{a_4^2})\phi$$
$$+(-c_{4,22}\frac{a_{\lambda,1}}{a_4} - c_{4,2}\frac{a_{\lambda,12}}{a_4})\bar{\phi}.$$

Hence we can directly calculate $dd^c(\sum |c_{\lambda,2}|^2)$ to prove the formula. q.e.d.

The following Proposition 7.8 is an extension of Proposition 6.2 to deal with the scalars $\{c_\lambda\}$, which will be useful for the further caluculations.

Proposition 7.8. *Let M be a complete connected 2-dimensional Riemannian manifold and $x : \mathrm{M} \longrightarrow \mathbf{X}$ an isometric minimal immersion of constant Kaehler function $\cos(\alpha)$, which is neither complex nor totally real. If $\cos(\alpha_2) \neq \pm 1$ on M and if K is strictly positive on M (hence M is compact), then we have $|c_{4,2}|^2 = 0$ on M.*

Proof. By (7.2), (7.3), Lemma 7.2 and Lemma 7.7, we have

$$\Delta(a_4^2|c_{4,2}|^2) = 10\mathrm{K}a_4^2|c_{4,2}|^2 + 4|a_4 c_{4,22} + \bar{a}_{4,1}c_{4,2}|^2,$$

which shows that $a_4^2|c_{4,2}|^2$ is constant. Hence we get $|c_{4,2}|^2 = 0$. q.e.d.

In order to express the equation $\Delta(c_3^2 \sum |c_{\lambda,2}|^2)$ in terms of K and $\cos(\alpha)$, we need to know further information on the third fundamental form of the immersion x. Let $H^{(2)}(t) = h_{t,111} + ih_{t,112}$ with $t = 9, 10, \cdots, 2n$, and put $H^{(2)} = \sum_t (H^{(2)}(t))^2$. We get $H^{(2)} = 4\sum \bar{a}_{\lambda,1}c_{\lambda,2}$ by (7.8), where λ runs from 5 through n. $|H^{(2)}|^2$ is a globally defined smooth function on $\mathbf{M}$, whose geometric meaning follows from the identity:

$$| H^{(2)} |^2 = (\| V_1^{(2)} \|^2 - \| V_2^{(2)} \|^2)^2 + 4 < V_1^{(2)}, V_2^{(2)} >^2 .$$

By (7.3), (7.4) and Proposition 7.8, we have

$$dH^{(2)} + 6iH^{(2)}\tilde{\theta}_{12} = 4\sum(\bar{a}_{\lambda,1}c_{\lambda,22} + \bar{a}_{\lambda,11}c_{\lambda,2})\bar{\phi},$$

since $\sum(\bar{a}_{\lambda,1}c_{\lambda,21} + \bar{a}_{\lambda,12}c_{\lambda,2}) = 0$, which is proved by using (7.3) and Proposition 7.8. By the same calculation as in the proof of Proposition 6.3, we have the following:

Proposition 7.9. *Under the same assumptions as in Proposition* 7.8, *we have* $H^{(2)} = 0$ *on* $\mathbf{M}$.

The smooth function $c_3^2 \sum |c_{\lambda,2}|^2$ is independent of the choice of normal vectors $\tilde{e}_t, 5 \le t \le 2n$. By Lemmas 7.2 and 7.7 as well as Propositions 7.8 and 7.9, we have

$$\Delta(c_3^2 \sum |c_{\lambda,2}|^2) = 2c_3^2 \sum |c_{\lambda,2}|^2\{6\mathrm{K} - \rho + 7\rho\cos(\alpha)\} + 4\sum |c_3 c_{\lambda,22} + c_{3,2}c_{\lambda,2}|^2, \tag{7.10}$$

from which we obtain:

Theorem 7.10. *Let* $\mathbf{X}$ *be a Kaehler manifold of complex dimension* n *of positive constant holomorphic sectional curvature* 4ρ *and* $\mathbf{M}$ *a complete connected* *oriented 2-dimensional Riemannian manifold. Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be a full isometric minimal immersion with constant Kaehler function , which is neither complex nor totally real. If the Gaussian curvature* K *of* $\mathbf{M}$ *is strictly positive and satisfies*

$$\mathrm{K} \geq \frac{(1-7\cos(\alpha))\rho}{6},$$

then K *is constant. When* $\mathbf{X}$ *is* $\mathbf{P}^n(\mathbf{C})$*, such an immersion* x *is congruent to either* $\varphi_{n,1}$, $\varphi_{n,2}$ *or* $\varphi_{n,3}$.

Proof. By Theorem 7.4 we may assume that there exists an open subset U such that $\mathrm{K} < (1-5\cos(\alpha))\rho/3$ on U. By Proposition 7.6 we get $\sum |c_\lambda|^2 \neq 0$ and $\sum |c_{\lambda,2}|^2 \neq 0$ at a point of U. By the assumption we have $\cos(\alpha) \neq \pm 1$ on $\mathbf{M}$. By (7.8) we have $6\mathrm{K} - \rho + 7\rho\cos(\alpha) = 0$, which shows that K is constant. q.e.d.

Corollary 7.11. *Under the same assumption as in Theorem* 7.10, *if*

$$\frac{(1-5\cos(\alpha))\rho}{3} > \mathrm{K} \geq \frac{(1-7\cos(\alpha))\rho}{6},$$

then K *is identically equal to* $(1-7\cos(\alpha))\rho/6$.

We know that the Gaussian curvature K and the Kaehler function $\cos(\alpha)$ of the generalized Veronese immersion $\varphi_{n,3}$ in $\mathbf{P}^n(\mathbf{C})$ satisfy $\mathrm{K} = (1-7\cos(\alpha))\rho/6$. Theorem 7.10 gives us a characterization of the generalized Veronese surfaces $\varphi_{n,1}$, $\varphi_{n,2}$ and $\varphi_{n,3}$ in $\mathbf{P}^n(\mathbf{C})$ by the pinched Gaussian curvature. Corollary 7.11 is an affirmative answer of the complex-version of Simon's conjecture under the condition that the Kaehler function is constant. In the following sections, we will consider further generalizations of Theorem 7.10 and Corollary 7.11.

8. J-invariant higher order osculating spaces

In Section 1 we defined the notions of the m-th osculating space and the m-th normal space of an immersed surface in an n-dimensional Riemannian space. In this section we introduce the concept of the *J-invariant m-th osculating space* of an immersed surface in a Kaehler manifold and calculate its *higher order fundamental forms* in terms of certain scalars locally defined on the surface. Let $\mathbf{X}$ be a Kaehler manifold of complex dimension n of constant holomorphic sectional curvature 4ρ and $x : \mathbf{M} \longrightarrow \mathbf{X}$ an isometric immersion of an oriented 2-dimensional Riemannian manifold $\mathbf{M}$ into $\mathbf{X}$. Let $C(t)$ be a smooth curve in $\mathbf{M}$ through a point $p = C(0)$ of $\mathbf{M}$ with parameter t, $|t| < L$. We denote by D^kC/dt^k the k-th covariant derivative along $C(t)$ in $\mathbf{X}$. Let $T^{(k)}_{J,p}(C)$ be a subspace of $T_p(\mathbf{X})$ spanned by

$$\{\frac{DC}{dt}, J\frac{DC}{dt}, \dots, \frac{D^kC}{dt^k}, J\frac{D^kC}{dt^k}\},$$

at $t = 0$, where J is the complex structure of $\mathbf{X}$. $T^{(k)}_{J,p}$ is defined to be the subspace spanned by all $T^{(k)}_{J,p}(C)$ for curves C lying on $\mathbf{M}$ through p and is called the *J-invariant k-th osculating space* of $\mathbf{M}$ at p . We then have

$$T_p(\mathbf{M}) \subset T^{(1)}_{J,p} \subset \cdots \subset T^{(m)}_{J,p} \subset T_p(\mathbf{X}).$$

Let $O^{(k+1)}_{J,p}$ be the orthogonal complement of $T^{(k)}_{J,p}$ in $T^{(k+1)}_{J,p}$ and $N^{(m)}_{J,p}$ the orthogonal complement of $T^{(m)}_{J,p}$ in $T_p(\mathbf{X})$, so that we have

$$T^{(k+1)}_{J,p} = T^{(k)}_{J,p} \oplus O^{(k+1)}_{J,p} \quad \text{and} \quad T_p(\mathbf{X}) = T^{(m)}_{J,p} \oplus N^{(m)}_{J,p}.$$

We put $O^{(1)}_{J,p} = T^{(1)}_{J,p}$. Note that $0 \le dim(O^{(k)}_{J,p}) \le 4$ and, if $dim(O^{(k)}_{J,p}) = 0$ for some k, then we have $dim(O^{(r)}_{J,p}) = 0$ for all $r \ge k$.

A point $p \in \mathbf{M}$ is called a *J-regular point of order m* if the J-invariant m-th osculating space $T^{(m)}_{J,p'}$ exists in a neighbourhood U of p and if each $O^{(k)}_{J,p'}$ is of dimension 4 for any $p' \in \mathbf{M}$ and $k = 1, 2, \dots, m$. We denote

$$O^{(k)}_J = \bigcup_{p \in \mathbf{M}} O^{(k)}_{J,p}.$$

We say that $x(\mathbf{M})$ is a *J-regular manifold* if each $O_J^{(k)}$ is of constant rank on $\mathbf{M}$ for any k. Note that $\operatorname{rank}(O_J^{(1)}) = 4$ if and only if x is neither holomorphic nor anti-holomorphic on $\mathbf{M}$.

Let $p \in \mathbf{M}$ be a J-regular point of order m. Then we have an orthogonal decomposition of $T_p(\mathbf{X})$ such that

$$T_p(\mathbf{X}) = O_{J,p}^{(1)} \oplus \cdots \oplus O_{J,p}^{(m)} \oplus N_{J,p}^{(m)}.$$

We define a J-canonical basis in $O_{J,p}^{(k)}$ as follows: Let $\{\tilde{e}_1, \tilde{e}_2\}$ be a local orthonormal frame of $\mathbf{M}$ and $\{\tilde{e}_{4k-3}, \tilde{e}_{4k-2}\}$ an orthonormal system of local normal vector fields along $\mathbf{M}$ such that it belongs to $O_{J,p}^{(k)}$ at p ($k \geq 2$). We put

$$\cos(\alpha) = < J\tilde{e}_1, \tilde{e}_2 > \quad \text{and} \quad \cos(\alpha_k) = < J\tilde{e}_{4k-3}, \tilde{e}_{4k-2} > . \tag{8.1}$$

So we have $\cos(\alpha) \neq \pm 1$ and $\cos(\alpha_k) \neq \pm 1$. As a consequence, we can define normal local vector fields $\tilde{e}_{4k-1}$ and $\tilde{e}_{4k}$ along $\mathbf{M}$ such that $\{\tilde{e}_{4k-3}, \tilde{e}_{4k-2}, \tilde{e}_{4k-1}, \tilde{e}_{4k}\}$ at p is an orthonormal basis of $O_{J,p}^{(k)}$ in the following way:

$$\begin{aligned} \tilde{e}_{4k-1} &= -\cot(\alpha_k)\tilde{e}_{4k-3} - \operatorname{cosec}(\alpha_k) J\tilde{e}_{4k-2}, \\ \tilde{e}_{4k} &= \operatorname{cosec}(\alpha_k) J\tilde{e}_{4k-3} - \cot(\alpha_k)\tilde{e}_{4k-2}. \end{aligned} \tag{8.2}$$

We denote the coframe field dual to this frame by $\{\tilde{\theta}_{4k-3}, \tilde{\theta}_{4k-2}, \tilde{\theta}_{4k-1}, \tilde{\theta}_{4k}\}$. We define vector fields $e_{4k-3}, e_{4k-2}, e_{4k-1}$ and e_{4k}, $k = 1, 2, \ldots, m$, in a neighbourhood of p as follows:

$$\begin{aligned} e_{4k-3} &= \cos(\frac{\alpha_k}{2})\tilde{e}_{4k-3} + \sin(\frac{\alpha_k}{2})\tilde{e}_{4k-1}, \\ e_{4k-2} &= \cos(\frac{\alpha_k}{2})\tilde{e}_{4k-2} + \sin(\frac{\alpha_k}{2})\tilde{e}_{4k}, \\ e_{4k-1} &= \sin(\frac{\alpha_k}{2})\tilde{e}_{4k-3} - \cos(\frac{\alpha_k}{2})\tilde{e}_{4k-1}, \\ e_{4k} &= -\sin(\frac{\alpha_k}{2})\tilde{e}_{4k-2} + \cos(\frac{\alpha_k}{2})\tilde{e}_{4k}, \end{aligned} \tag{8.3}$$

where we put $\alpha_1 = \alpha$. Then $\{e_{4k-3}, e_{4k-2}, e_{4k-1}, e_{4k}\}$ at p is a J-canonical basis of $O^{(k)}_{J,p}$, that is, an orthonormal basis of $O^{(k)}_{J,p}$ satisfying

$$Je_{4k-3} = e_{4k-2} \quad \text{and} \quad Je_{4k-1} = e_{4k}.$$

We denote the coframe field dual to this frame by $\{\theta_{4k-3}, \theta_{4k-2}, \theta_{4k-1}, \theta_{4k}\}$. Let $\{e_{4m+1}, \ldots, e_n\}$ be an orthonormal system of normal vector fields along x which is a J-canonical basis of $N^{(m)}_{J,p}$ at p. Let $\{\theta_{4m+1}, \ldots, \theta_n\}$ denote the coframe field dual to this frame. For $\alpha \geq 2m+1$ we put $\tilde{e}_{2\alpha-1} = e_{2\alpha-1}$ and $\tilde{e}_{2\alpha} = e_{2\alpha}$, so that we have $\tilde{\theta}_{2\alpha-1} = \theta_{2\alpha-1}$ and $\tilde{\theta}_{2\alpha} = \theta_{2\alpha}$. We put $\omega_\alpha = \theta_{2\alpha-1} + i\theta_{2\alpha}$, which is a local field of unitary coframes on $\mathbf{X}$. We then have, by (8.3):

$$\begin{aligned}
&\tilde{\theta}_{4k-3} + i\tilde{\theta}_{4k-2} = \cos(\frac{\alpha_k}{2})\omega_{2k-1} + \sin(\frac{\alpha_k}{2})\bar{\omega}_{2k}, \\
&\tilde{\theta}_{4k-1} + i\tilde{\theta}_{4k} = \sin(\frac{\alpha_k}{2})\omega_{2k-1} - \cos(\frac{\alpha_k}{2})\bar{\omega}_{2k}, \quad (k = 1, \ldots, m), \\
&\tilde{\theta}_{2\alpha-1} + i\tilde{\theta}_{2\alpha} = \theta_{2\alpha-1} + i\theta_{2\alpha} = \omega_\alpha, \quad (\alpha = 2m+1, \ldots, n).
\end{aligned} \tag{8.4}$$

Now we inductively introduce the higher order fundamental form

$$\sum h_{\lambda_k i_1 \cdots i_k} \tilde{\theta}_{i_1} \otimes \cdots \otimes \tilde{\theta}_{i_k} \otimes \tilde{e}_{\lambda_k}$$

of $x(\mathrm{M})$ in $\mathbf{X}$. Let $\{\tilde{\theta}_{AB}\}$ be the Riemannian connection form of $\mathbf{X}$ with respect to the canonical 1-form $\{\tilde{\theta}_A\}$ and $\{\omega_{\alpha\beta}\}$ the unitary connection form of $\mathbf{X}$ with respect to $\{\omega_\alpha\}$. We shall use the following ranges of indeces:

$$\begin{aligned}
&1 \leq A, B, \ldots \leq 2n, \quad 1 \leq i, j, \ldots \leq 2, \quad 3 \leq \lambda_0, \mu_0, \ldots \leq 4, \\
&4k-3 \leq \lambda_k, \mu_k, \ldots \leq 4k, \quad 4k+1 \leq s_k, t_k, \ldots \leq 2n, \\
&2k+1 \leq \alpha_k, \beta_k, \ldots \leq n, \quad \text{for } k = 1, 2, \ldots, m, \\
&4m+1 \leq \alpha, \beta, \ldots \leq 2n, \quad 2m+1 \leq \lambda, \mu, \ldots \leq n.
\end{aligned} \tag{8.5}$$

We denote by the same letters the restriction of the forms on **X** to **M**. We then have

$$
\begin{aligned}
&\tilde{\theta}_{\lambda_0} = \tilde{\theta}_{\lambda_k} = 0, \ (k \geq 2),\\
(8.6) \qquad &\tilde{\theta}_{\lambda_k \lambda_{l+2}} = 0, \ k = 1, 2, \ldots, m-2; \ l = k, \ldots, m-2,\\
&\tilde{\theta}_{\lambda_k \alpha} = 0, \ k = 1, 2, \ldots, m-1.
\end{aligned}
$$

By the exterior differentiation of (8.6) and the Riemannian structure equations (1.1), we get

$$
\begin{aligned}
&\sum_i \tilde{\theta}_i \wedge \tilde{\theta}_{i\lambda_o} = \sum_i \tilde{\theta}_i \wedge \tilde{\theta}_{i\lambda_2} = 0,\\
(8.7) \qquad &\sum_{\lambda_{k+1}} \tilde{\theta}_{\lambda_k \lambda_{k+1}} \wedge \tilde{\theta}_{\lambda_{k+1} \lambda_{k+2}} = 0, \ k = 2, \ldots, m-2,\\
&\sum_{\lambda_m} \tilde{\theta}_{\lambda_{m-1} \lambda_m} \wedge \tilde{\theta}_{\lambda_m \alpha} = 0.
\end{aligned}
$$

From these we can define inductively the quantities $h_{\lambda_k i_1 \cdots i_k}$ in the following way.

$$
\begin{aligned}
&\tilde{\theta}_{i\lambda_0} = \sum_j h_{\lambda_0 ij} \tilde{\theta}_j, \ \tilde{\theta}_{i\lambda_2} = \sum_j h_{\lambda_2 ij} \tilde{\theta}_j,\\
(8.8) \qquad &\sum_{\lambda_k} h_{\lambda_k i_1 \cdots i_k} \tilde{\theta}_{\lambda_k \lambda_{k+1}} = \sum_{i_{k+1}} h_{\lambda_{k+1} i_1 \cdots i_k i_{k+1}} \tilde{\theta}_{i_{k+1}},\\
&\sum_{\lambda_m} h_{\lambda_m i_1 \cdots i_m} \tilde{\theta}_{\lambda_m \alpha} = \sum_{i_{m+1}} h_{\alpha i_1 \cdots i_{m+1}} \tilde{\theta}_{i_{m+1}}.
\end{aligned}
$$

They satisfy the properties (1.7). Also the vectors

$$
V_1^{(k)} = \sum_{\lambda_k} h_{\lambda_k 1[k-1]1} \tilde{e}_{\lambda_k} \quad \text{and} \quad V_2^{(k)} = \sum_{\lambda_k} h_{\lambda_k 1[k-1]2} \tilde{e}_{\lambda_k}
$$

are elements of $O_{J,p}^{(k)}$ at $p \in$M for $k = 2, 3, \ldots, m$. We define

$$
(8.9) \qquad V_1^{(m+1)} = \sum_\alpha h_{\alpha 1[m]1} \tilde{e}_\alpha \quad \text{and} \quad V_2^{(m+1)} = \sum_\alpha h_{\alpha 1[m]2} \tilde{e}_\alpha,
$$

which are called the (m+1)-*th normal vectors* at a J-regular point of order m.

9. J-regular points of order m

In this section, adopting the normalized k-th normal vectors as a basis of $O^{(k)}_{J,p}$ for $k = 2, \cdots, m$, we calculate the $(m+1)$*-th fundamental forms*, the $(m+1)$*-th normal vectors* in terms of some complex-valued smooth functions defined locally on M and study their properties. In the case of $m = 2$ these computations are accomplished in Sections 6 and 7.

Before computing the $(m+1)$-th fundamental form, we recall our assumptions on the immersion. Let M be a complete connected oriented 2-dimensional Riemannian manifold such that the Gaussian curvature K of M satisfies $K \geq \delta > 0$ for some positive number δ and $x : \mathbf{M} \longrightarrow \mathbf{X}$ an isometric minimal immersion with constant Kaehler function $\cos(\alpha)$. We assume that every point of M is J-regular of order $m(\geq 3)$ and that the k-th normal vectors $V^{(k)}_1$ and $V^{(k)}_2$ are perpendicular to each other and have the same non-zero length for $k = 3, \cdots, m$. Normalizing these vectors we adopt them as a basis of $O^{(k)}_{J,p}$, so that we have

$$\tilde{e}_{4k-3} = \frac{V^{(k)}_1}{\| V^{(k)}_1 \|}, \quad \tilde{e}_{4k-2} = \frac{V^{(k)}_2}{\| V^{(k)}_2 \|}$$

and

$$\cos(\alpha_k) = < J\tilde{e}_{4k-3}, \tilde{e}_{4k-2} > \neq \pm 1$$

on M. With respect to these frames we assume

$$h_{4k-3,1[k-1]1} = -h_{4k-2,1[k-1]2}, \tag{9.1}$$

$$h_{4k-3,1[k-1]2} = h_{4k-2,1[k-1]1} = h_{t_k,1[k-1]1} = h_{t_k,1[k-1]2} = 0, \quad (t_k \geq 4k-1).$$

We put

$$c_{2k-1,2[k-2]} = -\cos(\frac{\alpha_k}{2})h_{4k-3,1[k-1]1},$$

$$a_{2k,1[k-2]} = -\sin(\frac{\alpha_k}{2})h_{4k-3,1[k-1]1}, \tag{9.2}$$

$$c_{2k,2[k-2]} = a_{2k-1,1[k-2]} = c_{\lambda_k,2[k-2]} = a_{\lambda_k,1[k-2]} = 0, \quad (\lambda_k \geq 2k+1),$$

where $c_{2k-1,2[k-1]}$ and others are real-valued smooth functions locally defined on M. We assume that they satisfy the following:

$$\begin{aligned}
&c_{2k-3,2[k-3]}\omega_{2k-1,2k-3} = -c_{2k-1,2[k-2]}\bar{\phi},\\
&a_{2k-2,1[k-3]}\omega_{2k,2k-2} = -a_{2k,1[k-2]}\phi,\\
&\omega_{2k,2k-3} = \omega_{2k-1,2k-2} = \omega_{\lambda_k,2k-3} = \omega_{\lambda_k,2k-2} = 0 \ (\lambda_k \geq 2k+1),\\
&dc_{2k-1,2[k-2]} + ikc_{2k-1,2[k-2]}\tilde{\theta}_{12} - c_{2k-1,2[k-2]}\omega_{2k-1,2k-1} = c_{2k-1,2[k-1]}\bar{\phi},\\
&da_{2k,1[k-2]} - ika_{2k,1[k-2]}\tilde{\theta}_{12} - a_{2k,1[k-2]}\omega_{2k,2k} = a_{2k,1[k-1]}\phi,\\
&c_{2k-1,2[k-2]}\omega_{2k,2k-1} = -c_{2k,2[k-1]}\bar{\phi},\\
&a_{2k,1[k-2]}\omega_{2k-1,2k} = -a_{2k-1,1[k-1]}\phi,\\
&c_{2k-1,2[k-2]}\omega_{\lambda_k,2k-1} = -c_{\lambda_k,2[k-1]}\bar{\phi},\\
&a_{2k,1[k-2]}\omega_{\lambda_k,2k} = -a_{\lambda_k,1[k-1]}\phi, \quad (\lambda_k \geq 2k+1), \quad \text{for } k = 3, \cdots, m.
\end{aligned} \tag{9.3}$$

By (9.3) we have

$$\begin{aligned}
\Delta(c_{2k-1,2[k-2]})^2 &= 2k\mathrm{K}(c_{2k-1,2[k-2]})^2 + 4\{(c_{2k-1,2[k-1]})^2 + (c_{2k+1,2[k-2]})^2\}\\
&\quad + 4\rho(c_{2k-1,2[k-2]})^2\cos(\alpha) - \frac{4(c_{2k-1,2[k-2]})^4}{(c_{2k-3,2[k-3]})^2},\\
&\qquad \text{for } k = 3, \cdots, m-1,
\end{aligned}$$

$$\begin{aligned}
\Delta(c_{2m-1,2[m-2]})^2 &= 2m\mathrm{K}(c_{2m-1,2[m-2]})^2 + 4\sum_{\lambda \geq 2m-1}|c_{\lambda,2[m-1]}|^2\\
&\quad + 4\rho(c_{2m-1,2[m-2]})^2\cos(\alpha) - \frac{4(c_{2m-1,2[m-2]})^4}{(c_{2m-3,2[m-3]})^2},\\
\Delta(a_{2m,1[m-2]})^2 &= 2m\mathrm{K}(a_{2m,1[m-2]})^2 + 4\sum_{\lambda \geq 2m-1}|a_{\lambda,1[m-1]}|^2\\
&\quad - 4\rho(a_{2m,1[m-2]})^2\cos(\alpha) - \frac{4(a_{2m,1[m-2]})^4}{(a_{2m-2,2[m-3]})^2}.
\end{aligned} \tag{9.4}$$

Now we calculate the $(m+1)$-th normal vectors. Using the third equality in (8.8) and (9.1), we have, for $\lambda \geq 2m+1$,

$$
\begin{aligned}
&h_{4m-3,1[m]}\tilde{\theta}_{4m-3,2\lambda-1} = h_{2\lambda-1,1[m]1}\tilde{\theta}_1 + h_{2\lambda-1,1[m]2}\tilde{\theta}_2,\\
(9.5)\qquad &\\
&h_{4m-3,1[m]}\tilde{\theta}_{4m-3,2\lambda} = h_{2\lambda,1[m]1}\tilde{\theta}_1 + h_{2\lambda,1[m]2}\tilde{\theta}_2.
\end{aligned}
$$

By taking the exterior derivative of (8.4) and using the structure equations (5.1) of $\mathbf{X}$, we get, for $k = 1, 2, \cdots, m$:

$$
\begin{aligned}
&\tilde{\theta}_{4k-3,2\lambda-1} + i\tilde{\theta}_{4k-2,2\lambda-1} = \cos(\frac{\alpha_k}{2})\omega_{2k-1,\lambda} + \sin(\frac{\alpha_k}{2})\bar{\omega}_{2k,\lambda},\\
&\tilde{\theta}_{4k-3,2\lambda} + i\tilde{\theta}_{4k-2,2\lambda} = i\{\cos(\frac{\alpha_k}{2})\omega_{2k-1,\lambda} - \sin(\frac{\alpha_k}{2})\bar{\omega}_{2k,\lambda}\},\\
(9.6)\qquad &\\
&\tilde{\theta}_{4k-1,2\lambda-1} + i\tilde{\theta}_{4k,2\lambda-1} = \sin(\frac{\alpha_k}{2})\omega_{2k-1,\lambda} + \cos(\frac{\alpha_k}{2})\bar{\omega}_{2k,\lambda},\\
&\tilde{\theta}_{4k-1,2\lambda} + i\tilde{\theta}_{4k,2\lambda} = i\{\sin(\frac{\alpha_k}{2})\omega_{2k-1,\lambda} + \cos(\frac{\alpha_k}{2})\bar{\omega}_{2k,\lambda}\}.
\end{aligned}
$$

Substituting (9.1), (9.2), the eighth and the ninth equalities in (9.3) and (9.6) into (9.5), we get

$$
\begin{aligned}
&h_{2\lambda-1,1[m]1} = -\frac{1}{2}(a_{\lambda,1[m-1]} + \bar{a}_{\lambda,1[m-1]} + c_{\lambda,2[m-1]} + \bar{c}_{\lambda,2[m-1]}),\\
&h_{2\lambda-1,1[m]2} = -\frac{i}{2}(a_{\lambda,1[m-1]} - \bar{a}_{\lambda,1[m-1]} - c_{\lambda,2[m-1]} + \bar{c}_{\lambda,2[m-1]}),\\
(9.7)\qquad &\\
&h_{2\lambda,1[m]1} = \frac{i}{2}(a_{\lambda,1[m-1]} - \bar{a}_{\lambda,1[m-1]} + c_{\lambda,2[m-1]} - \bar{c}_{\lambda,2[m-1]}),\\
&h_{2\lambda,1[m]2} = -\frac{1}{2}(a_{\lambda,1[m-1]} + \bar{a}_{\lambda,1[m-1]} - c_{\lambda,2[m-1]} - \bar{c}_{\lambda,2[m-1]}).
\end{aligned}
$$

By taking the exterior derivatives of the sixth through the ninth equalities in (9.3), we have

$$dc_{2m,2[m-1]} + i(m+1)c_{2m,2[m-1]}\tilde{\theta}_{12} - c_{2m,2[m-1]}\omega_{2m,2m} = c_{2m,2[m]}\bar{\phi},$$

$$dc_{\lambda,2[m-1]} + i(m+1)c_{\lambda,2[m-1]}\tilde{\theta}_{12} - \sum_{\mu} c_{\mu,2[m-1]}\omega_{\lambda,\mu} = c_{\lambda,2[m-1]1}\phi + c_{\lambda,2[m]}\bar{\phi},$$

$$\text{with } c_{\lambda,2[m-1]1} = -\frac{a_{\lambda,1[m-1]}c_{2m,2[m-1]}}{a_{2m,1[m-2]}},$$

$$(9.8)\ da_{2m-1,1[m-1]} - i(m+1)a_{2m-1,1[m-1]}\tilde{\theta}_{12} - a_{2m-1,1[m-1]}\omega_{2m-1,2m-1} = a_{2m-1,1[m]}\phi,$$

$$da_{\lambda,1[m-1]} - i(m+1)a_{\lambda,1[m-1]}\tilde{\theta}_{12} - \sum_{\mu} a_{\mu,1[m-1]}\omega_{\lambda,\mu} = a_{\lambda,1[m]}\phi + a_{\lambda,1[m-1]2}\bar{\phi},$$

$$\text{with } a_{\lambda,1[m-1]2} = -\frac{c_{\lambda,2[m-1]}a_{2m-1,1[m-1]}}{c_{2m-1,2[m-2]}},$$

where $c_{2m,2[m]}$, $c_{\lambda,2[m]}$, $a_{2m-1,1[m]}$ and $a_{\lambda,1[m]}$ are complex-valued smooth functions defined locally on $\mathbf{M}$.

The following Proposition 9.1, which is a generalization of Proposition 7.8, is essentially used in the calculation of the Laplacians of certain scalars defined on $\mathbf{M}$ (cf.Lemma 9.3).

Proposition 9.1. *Let* $\mathbf{X}$ *be a Kaehler manifold of complex dimension* n *of positive constant holomorphic sectional curvature* 4ρ *and* $\mathbf{M}$ *a complete connected oriented* 2*-dimensional Riemannian manifold such that the Gaussian curvature* K *of* $\mathbf{M}$ *satisfies* $\mathrm{K} \geq \delta > 0$ *for some positive number* δ. *Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be an isometric minimal immersion of* $\mathbf{M}$ *into* $\mathbf{X}$ *with constant Kaehler function* $\cos(\alpha)$, *which is neither complex nor totally real. We assume that each point of* $\mathbf{M}$ *is a J-regular point of order* m, *and formulas* (9.1), ..., (9.8) *in Section* 9 *are valid on* $\mathbf{M}$. *Then we have*

$$c_{2m,2[m-1]} = 0.$$

Proof. Using the first equality in (9.8), we have

$$d|c_{2m,2[m-1]}|^2 = c_{2m,2[m-1]}\bar{c}_{2m,2[m]}\phi + \bar{c}_{2m,2[m-1]}c_{2m,2[m]}\bar{\phi},$$

$$\begin{aligned}\Delta|c_{2m,2[m-1]}|^2 &= 2(m+1)\mathrm{K}|c_{2m,2[m-1]}|^2 + 4|c_{2m,2[m]}|^2\\ &+ 4|c_{2m,2[m-1]}|^2\Big\{\frac{(a_{2m,1[m-2]})^2}{(a_{2m-2,1[m-3]})^2} - \frac{|c_{2m,2[m-1]}|^2}{(c_{2m-1,2[m-2]})^2}\\ &- \frac{\sum_{\mu\geq 2m+1}|a_{\mu,1[m-1]}|^2}{(a_{2m,1[m-2]})^2} + \rho\cos(\alpha)\Big\}.\end{aligned}$$

Combining the third equality in (9.4) and the above equalities, we get

$$\triangle\{(a_{2m,1[m-2]})^2|c_{2m,2[m-1]}|^2\} = 2(2m+1)\mathrm{K}(a_{2m,1[m-2]})^2|c_{2m,2[m-1]}|^2$$

$$+\, 4|a_{2m,1[m-2]}c_{2m,2[m]}+a_{2m,1[m-1]}c_{2m,2[m-1]}|^2.$$

By the assumption we see that $\mathbf{M}$ is compact and $a_{2m,1[m-2]} \neq 0$ on $\mathbf{M}$. Using the above formula and the maximum principle, we find $c_{2m,2[m-1]} = 0$. q.e.d.

Let $V_1^{(m+1)}$ and $V_2^{(m+1)}$ be the $(m+1)$-th normal vectors of $N_p^{(m)}$ at $p \in \mathbf{M}$, which are defined by (8.9). We put

$$\Theta_{(m+1)} = \{p \in \mathbf{M};\, V_1^{(m+1)}(p) = 0 \ \text{ or } \ V_2^{(m+1)}(p) = 0\}. \tag{9.9}$$

Using (9.7) we have

$$\sum_\lambda (a_{\lambda,1[m-1]} - c_{\lambda,2[m-1]})^2 = 0 \ \text{ or } \ \sum_\lambda (a_{\lambda,1[m-1]} + c_{\lambda,2[m-1]})^2 = 0 \tag{9.10}$$

at $p \in \Theta_{(m+1)}$. Next we define, at $p \in \mathbf{M}\setminus\Theta_{(m+1)}$,

$$\cos(\alpha_{m+1}) =< \frac{JV_1^{(m+1)}}{\| JV_1^{(m+1)} \|}, \frac{V_2^{(m+1)}}{\| V_2^{(m+1)} \|} >, \tag{9.11}$$

which is also expressed by (9.7) as

$$\cos(\alpha_{m+1}) = \frac{\sum_\lambda\{|a_{\lambda,1[m-1]}|^2 - |c_{\lambda,2[m-1]}|^2\}}{\sum_\lambda\{|a_{\lambda,1[m-1]}|^2 + |c_{\lambda,2[m-1]}|^2\}}. \tag{9.12}$$

Using the third equality in (1.7), we see that $O_{J,p}^{(m+1)}$ is spanned by $V_1^{(m+1)}$, $V_2^{(m+1)}$, $JV_1^{(m+1)}$ and $JV_2^{(m+1)}$ at $p \in \mathbf{M}$. If $\cos(\alpha_{m+1}) = 1$ or -1, we have $\dim(O_{J,p}^{(m+1)}) < 4$. Hence, if we assume that each point of $\mathbf{M}$ is a J-regular point of order $(m+1)$, then $\Theta_{(m+1)} = \emptyset$ and $\cos(\alpha_{m+1})$ does not attain $0, 1$ or -1 on $\mathbf{M}$.

Now we define $H_{2\lambda-1}^{(m+1)}$ and $H_{2\lambda}^{(m+1)}$ by

$$V_1^{(m+1)} + iV_2^{(m+1)} = \sum_\lambda (H_{2\lambda-1}^{(m+1)} e_{2\lambda-1} + H_{2\lambda}^{(m+1)} e_{2\lambda}) \tag{9.13}$$

and put

$$H^{(m+1)} = \sum_{\lambda}\{(H^{(m+1)}_{2\lambda-1})^2 + (H^{(m+1)}_{2\lambda})^2\}. \tag{9.14}$$

Using (9.7) we have

$$H^{(m+1)} = 4\sum_{\lambda} \bar{a}_{\lambda,1[m-1]}c_{\lambda,2[m-1]}. \tag{9.15}$$

Note that $|H^{(m+1)}|^2$ is a globally defined smooth function on $\mathbf{M}$. The geometric meaning of $|H^{(m+1)}|^2$ becomes clear from the identity

$$|H^{(m+1)}|^2 = (\| V_1^{(m+1)} \|^2 - \| V_2^{(m+1)} \|^2)^2 + 4 < V_1^{(m+1)}, V_2^{(m+1)} >^2. \tag{9.16}$$

Proposition 9.2. *In addition to the assumptions of Proposition 9.1, we assume that each point of* $\mathbf{M}$ *is a J-regular point of order* $(m+1)$*. Then we have* $H^{(m+1)} = 0$ *on* $\mathbf{M}$.

Proof. Using Proposition 9.1 and (9.8), we have

$$dH^{(m+1)} + 2(m+1)iH^{(m+1)}\tilde{\theta}_{12} = 4\sum_{\lambda\geq 2m+1}(\bar{a}_{\lambda,1[m]}c_{\lambda,2[m-1]} + \bar{a}_{\lambda,1[m-1]}c_{\lambda,2[m]})\bar{\phi},$$

$$\triangle|H^{(m+1)}|^2 = 2\{2(m+1)\mathrm{K}|H^{(m+1)}|^2 + 2|\sum_{\lambda}(\bar{a}_{\lambda,1[m]}c_{\lambda,2[m-1]} + \bar{a}_{\lambda,1[m-1]}c_{\lambda,2[m]})|^2\}.$$

From these we can prove Proposition 9.2. q.e.d.

To investigate properties of the $(m+1)$-th fundamental forms of the immersion, it follows from (9.7) that it suffices to study the scalars $\{c_{\lambda,2[m-1]}, a_{\lambda,1[m-1]}\}$. From this point of view, we now caluculate the Laplacians of $(\sum_\lambda |c_{\lambda,2[m-1]}|^2)$ and $(\sum_\lambda |a_{\lambda,1[m-1]}|^2)$.

Lemma 9.3. *Let* $\mathbf{X}$ *be a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature* 4ρ *and* $\mathbf{M}$ *a complete connected oriented 2-dimensional Riemannian manifold such that the Gaussian curvature* K *of* $\mathbf{M}$

satiesfies K $\geq \delta > 0$ *for some positive number* δ. *Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be an isometric minimal immersion of* $\mathbf{M}$ *into* $\mathbf{X}$ *with constant Kaehler function* $\cos(\alpha)$, *which is neither complex nor totally real. We assume that each point of* $\mathbf{M}$ *is a J-regular point of order* $(m+1)$ *and formulas* $(9.1), \ldots, (9.8)$ *in Section* 9 *are valid on* $\mathbf{M}$. *Then we have*

$$\begin{aligned}
\Delta(\sum_\lambda |c_{\lambda,2[m-1]}|^2) &= 2(m+1)\mathrm{K}(\sum_\lambda |c_{\lambda,2[m-1]}|^2) \ + \ 4\sum_\lambda |c_{\lambda,2[m]}|^2 \\
&\quad - \frac{4(\sum_\lambda |c_{\lambda,2[m-1]}|^2)^2}{(c_{2m-1,2[m-2]})^2} + 4\rho\cos(\alpha)(\sum_\lambda |c_{\lambda,2[m-1]}|^2). \\
\Delta(\sum_\lambda |a_{\lambda,1[m-1]}|^2) &= 2(m+1)\mathrm{K}(\sum_\lambda |a_{\lambda,1[m-1]}|^2) \ + \ 4\sum_\lambda |a_{\lambda,2[m]}|^2 \\
&\quad - \frac{4(\sum_\lambda |a_{\lambda,1[m-1]}|^2)^2}{(a_{2m,1[m-2]})^2} - 4\rho\cos(\alpha)(\sum_\lambda |a_{\lambda,1[m-1]}|^2).
\end{aligned}$$

Proof. Using Proposition 9.1 and the second equality in (9.8), we have

$$d(\sum_\lambda |c_{\lambda,2[m-1]}|^2) = \sum_\lambda (c_{\lambda,2[m-1]}\bar{c}_{\lambda,2[m]}\phi \ + \ \bar{c}_{\lambda,2[m-1]}c_{\lambda,2[m]}\bar{\phi}).$$

This implies

$$\begin{aligned}
d^c(\sum_\lambda |c_{\lambda,2[m-1]}|^2) &= i\sum_\lambda(-\, c_{\lambda,2[m-1]}\bar{c}_{\lambda,2[m]}\phi + \bar{c}_{\lambda,2[m-1]}c_{\lambda,2[m]}\bar{\phi}) \\
&= i\sum_\lambda(-\, c_{\lambda,2[m-1]}d\bar{c}_{\lambda,2[m-1]} + \bar{c}_{\lambda,2[m-1]}dc_{\lambda,2[m-1]} \\
&\qquad + 2i(m+1)|c_{\lambda,2[m-1]}|^2\tilde{\theta}_{12} - 2\bar{c}_{\lambda,2[m-1]}\sum_\mu c_{\mu,2[m-1]}\omega_{\mu\lambda}),
\end{aligned}$$

where we used (9.8) and Proposition 9.1 once more. We calculate $dd^c(\sum |c_{\lambda,2[m-1]}|^2)$ by using the formulas obtained in this Section and get the first formula of Lemma 9.3. In a similar way we can prove the formula for $\Delta(\sum_\lambda |a_{\lambda,1[m-1]}|^2)$ by using the fourth equality in (9.8). q.e.d.

10. Curvature pinching theorems

In Section 9 we assumed that each point of M is a *J-regular point of order* m and that all formulas in Section 5 are valid. In this section we show that the same formulas as in Section 9 hold in a neighbourhood of a *J-regular point of order* (m+1).

We assume that $p \in \mathrm{M}$ is a J-regular point of order $(m+1)$. By Proposition 9.2 we have that $V_1^{(m+1)}$ and $V_2^{(m+1)}$ are perpendicular to each other and have the same length at $p \in \mathrm{M}$. Normalizing these vectors we adopt them as a basis of $O_{J,p'}^{(m+1)}$ in a neighbourhood of p, so that we have

$$\tilde{e}_{4m+1} = \frac{V_1^{(m+1)}}{\| V_1^{(m+1)} \|} \quad \text{and} \quad \tilde{e}_{4m+2} = \frac{V_2^{(m+1)}}{\| V_2^{(m+1)} \|}$$

and

$$\cos(\alpha_{m+1}) = < J\tilde{e}_{4m+1}, \tilde{e}_{4m+2} > \neq \pm 1.$$

With respect to these new frames, we have

$$\begin{aligned} &h_{4m+1,1[m]1} = -h_{4m+2,1[m]2} \ (\neq 0), \\ (10.1) \qquad & \\ &h_{4m+1,1[m]2} = h_{4m+2,1[m]1} = h_{\lambda,1[m]1} = h_{\lambda,1[m]2} = 0, \quad (\lambda \geq 4m{+}3). \end{aligned}$$

Substituting (10.1) into (9.5), we have

$$\begin{aligned} &h_{4m-3,1[m]}(\tilde{\theta}_{4m-3,4m+1} + i\tilde{\theta}_{4m-2,4m+1}) = h_{4m+1,1[m]1}\phi, \\ &h_{4m-3,1[m]}(\tilde{\theta}_{4m-3,4m+2} + i\tilde{\theta}_{4m-2,4m+2}) = -h_{4m+2,1[m]2}\phi, \\ (10.2) \qquad &h_{4m-3,1[m]}(\tilde{\theta}_{4m-3,4m+3} + i\tilde{\theta}_{4m-2,4m+3}) = 0, \\ &h_{4m-3,1[m]}(\tilde{\theta}_{4m-3,4m+4} + i\tilde{\theta}_{4m-2,4m+4}) = 0, \\ &h_{4m-3,1[m]}(\tilde{\theta}_{4m-3,2\alpha-1} + i\tilde{\theta}_{4m-2,2\alpha-1}) = 0, \\ &h_{4m-3,1[m]}(\tilde{\theta}_{4m-3,2\alpha} + i\tilde{\theta}_{4m-2,2\alpha}) = 0, \quad (\alpha \geq 2m{+}3). \end{aligned}$$

On the other hand, by taking the exterior derivatives of (8.4) for $k, l = 1, 2, \cdots, m+1$, we have

$$
\begin{aligned}
&\tilde{\theta}_{4k-3,4l-3} + i\tilde{\theta}_{4k-2,4l-3} \\
&\qquad = \cos(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\omega_{2k-1,2l-1} + \cos(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\omega_{2k-1,2l} \\
&\qquad + \sin(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l-1} + \sin(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l}, \\
&\tilde{\theta}_{4k-3,4l-2} + i\tilde{\theta}_{4k-2,4l-2} \\
&\qquad = i\{\cos(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\omega_{2k-1,2l-1} - \cos(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\omega_{2k-1,2l} \\
&\qquad - \sin(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l-1} + \sin(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l}\}, \\
&\tilde{\theta}_{4k-3,4l-1} + i\tilde{\theta}_{4k-2,4l-1} \\
&\qquad = \cos(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\omega_{2k-1,2l-1} - \cos(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\omega_{2k-1,2l} \\
&\qquad + \sin(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l-1} - \sin(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l}, \\
&\tilde{\theta}_{4k-3,4l} + i\tilde{\theta}_{4k-2,4l} \\
&\qquad = i\{\cos(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\omega_{2k-1,2l-1} + \cos(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\omega_{2k-1,2l} \\
&\qquad - \sin(\frac{\alpha_k}{2})\sin(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l-1} - \sin(\frac{\alpha_k}{2})\cos(\frac{\alpha_l}{2})\bar{\omega}_{2k,2l}\}.
\end{aligned}
\tag{10.3}
$$

In the first and second equalities in (6.2) and the eighth and ninth equalities in (9.3) we put $k = m$. Then we have

$$h_{4m-3,1[m]1} = -\sec(\frac{\alpha_m}{2})c_{2m-1,2[m-2]} = -\operatorname{cosec}(\frac{\alpha_m}{2})a_{2m,1[m-2]},$$

$$c_{2m-1,2[m-2]}\omega_{2m-1,\lambda} = \bar{c}_{\lambda,2[m-1]}\phi, \quad a_{2m,1[m-2]}\omega_{2m,\lambda} = \bar{a}_{\lambda,1[m-1]}\bar{\phi},$$

for $\lambda \geq 2m+1$. Substituting these equalities and (10.3) into (10.2), we get

$$
\begin{aligned}
\cos(\frac{\alpha_{m+1}}{2})(\bar{c}_{2m+1,2[m-1]} + a_{2m+1,1[m-1]}) + \sin(\frac{\alpha_{m+1}}{2})(\bar{c}_{2m+2,2[m-1]} + a_{2m+2,1[m-1]}) \\
= -h_{4m+1,1[m]1},
\end{aligned}
$$

$$\cos(\frac{\alpha_{m+1}}{2})(\bar{c}_{2m+1,2[m-1]} - a_{2m+1,1[m-1]}) - \sin(\frac{\alpha_{m+1}}{2})(\bar{c}_{2m+2,2[m-1]} - a_{2m+2,1[m-1]})$$

$$= h_{4m+2,1[m]2},$$

$$\sin(\frac{\alpha_{m+1}}{2})(\bar{c}_{2m+1,2[m-1]} + a_{2m+1,1[m-1]}) - \cos(\frac{\alpha_{m+1}}{2})(\bar{c}_{2m+2,2[m-1]} + a_{2m+2,1[m-1]})$$

$$= 0, \tag{10.4}$$

$$\sin(\frac{\alpha_{m+1}}{2})(-\bar{c}_{2m+1,2[m-1]} + a_{2m+1,1[m-1]}) + \cos(\frac{\alpha_{m+1}}{2})(-\bar{c}_{2m+2,2[m-1]} + a_{2m+2,1[m-1]})$$

$$= 0,$$

$$\bar{c}_{\lambda,2[m-1]} - a_{\lambda,1[m-1]} = 0,$$

$$\bar{c}_{\lambda,2[m-1]} + a_{\lambda,1[m-1]} = 0.$$

Solving the above equations, we have

$$\bar{c}_{2m+1,2[m-1]} = \cot(\frac{\alpha_{m+1}}{2})a_{2m+2,1[m-1]},$$

$$a_{2m+1,1[m-1]} = \cot(\frac{\alpha_{m+1}}{2})\bar{c}_{2m+2,2[m-1]}, \tag{10.5}$$

$$\bar{c}_{\lambda,2[m-1]} = a_{\lambda,1[m-1]} = 0.$$

Moreover, since $H^{(m+1)} = 0$, we see that $c_{2m+1,2[m-1]}$ is real-valued and $c_{2m+2,2[m-1]} = 0$. Summarizing up these results, we have

$$h_{4m+1,1[m]1} = -h_{4m+3,1[m]2} = -\sec(\frac{\alpha_{m+1}}{2})c_{2m+1,2[m-1]},$$

$$h_{4m+1,1[m]2} = h_{4m+2,1[m]1} = h_{t,1[m]1} = h_{t,1[m]2} = 0, \;\; (t \geq 4m+3), \tag{10.6}$$

$$c_{2m+1,2[m-1]} = \cot(\frac{\alpha_{m+1}}{2})a_{2m+1,1[m-1]},$$

$$c_{2m+2,2[m-1]} = a_{2m+1,1[m-1]} = c_{\lambda,2[m-1]} = a_{\lambda,1[m-1]} = 0, (\lambda \geq 2m+3).$$

Now substituting (10.6) into the eighth and ninth equalities in (9.3), we have

$$c_{2m-1,2[m-2]}\omega_{2m+1,2m-1} = -c_{2m+1,2[m-1]}\bar{\phi},$$

$$(10.7) \qquad a_{2m,1[m-2]}\omega_{2m+2,2m} = -a_{2m+2,1[m-1]}\phi,$$

$$\omega_{2m+2,2m-1} = \omega_{2m+1,2m} = \omega_{\alpha,2m-1} = \omega_{\alpha,2m} = 0, \quad (\alpha \geq 2m+3).$$

Moreover, by (9.8), we have

$$dc_{2m-1,2[m-1]} + i(m+1)c_{2m+1,2[m-1]}\tilde{\theta}_{12} - c_{2m+1,2[m-1]}\omega_{2m+1,2m+1}$$
$$= c_{2m+1,2[m]}\bar{\phi},$$

$$da_{2m+2,1[m-1]} - i(m+1)a_{2m+2,1[m-1]}\tilde{\theta}_{12} - a_{2m+2,1[m-1]}\omega_{2m+2,2m+2}$$
$$= a_{2m+2,1[m]}\phi,$$

$$(10.8) \qquad c_{2m+1,2[m-1]}\omega_{2m+2,2m+1} = -c_{2m+2,2[m]}\bar{\phi},$$

$$a_{2m+2,1[m-1]}\omega_{2m+1,2m+2} = -a_{2m+1,1[m]}\phi,$$

$$c_{2m+1,2[m-1]}\omega_{\lambda,2m+1} = -c_{\lambda,2[m]}\bar{\phi},$$

$$a_{2m+2,1[m-1]}\omega_{\lambda,2m+2} = -a_{\lambda,1[m]}\phi, \quad (\lambda \geq 2m+3).$$

The formulas (10.6), (10.7), (10.8) and Lemma 9.3 show us that (9.2), (9.3) and (9.4) are valid for $k = (m+1)$.

We define smooth functions on $\mathbf{M}$ by

$$(10.9) \qquad \mathcal{C}_k^2 = c_3^2 c_{5,2}^2 \cdots c_{2k-1,2[k-2]}^2, \quad k = 2, 3, \ldots, m.$$

Note that these functions are scalar invariants of the immersion x , which can be seen in a way similar to that in [34, p.372]. Using (9.2) and (9.3), we get

$$d\mathcal{C}_k^2 = \mathcal{C}_k(A_k\phi + \bar{A}_k\bar{\phi}),$$

where A_k satisfies

$$\bar{A}_k = \mathcal{C}_{k-1}c_{2k-1,2[k-1]} + \bar{A}_{k-1}c_{2k-1,2[k-2]}$$

for $k = 3, \ldots, m$ and $\bar{A}_2 = c_{3,2}$.

Using (9.4) and Lemma 9.3, we have

Lemma 10.1. *Let* $\mathbf{X}$ *be a Kaehler manifold of complex dimension n of positive constant holomorphic sectional curvature* 4ρ *and* $\mathbf{M}$ *a complete connected oriented 2-dimensional Riemannian manifold such that the Gaussian curvature* K *of* $\mathbf{M}$ *satiesfies* $\mathrm{K} \geq \delta > 0$ *for some positive number* δ*. Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be an isometric minimal immersion of* $\mathbf{M}$ *into* $\mathbf{X}$ *with constant Kaehler function* $\cos(\alpha)$*, which is neither complex nor totally real. We assume that each point of* $\mathbf{M}$ *is a J-regular point of order* $(m+1)$*. Then we have*

$$\begin{aligned}(10.10)\qquad \triangle \mathcal{C}_m^2 = 2\mathcal{C}_m^2\{\frac{m(m+1)\mathrm{K}}{2} \ - \ \rho \ + \ (2m+1)\rho\cos(\alpha)\} \\ + 4|A_m|^2 \ + \ 4\mathcal{C}_{m-1}^2\sum_\lambda |c_{\lambda,2[m-1]}|^2,\end{aligned}$$

$$\begin{aligned}(10.11)\qquad \triangle(\mathcal{C}_m^2\sum_\lambda |c_{\lambda,2[m-1]}|^2) = 2\mathcal{C}_m^2\sum_\lambda |c_{\lambda,2[m-1]}|^2\{\frac{(m+1)(m+2)\mathrm{K}}{2} \\ - \rho + (2m{+}3)\rho\cos(\alpha)\} \\ + 4\sum_\lambda |\mathcal{C}_m c_{\lambda,2[m]} + \bar{A}_m c_{\lambda,2[m-1]}|^2.\end{aligned}$$

These formulas (10.10) and (10.11) give us an important information on examining the properties of the pinched Gaussian curvature of a minimal surface in $\mathbf{P}^n(\mathbf{C})$. Note that (10.11) coinsides with (3.8) in [34] for $m = 2$.

Now we are in a position to study the conjecture by Bolton et al. [5] stated in Section 7, which is closely related to the problems of classifying minimal surfaces of constant Kaehler function in $\mathbf{P}^n(\mathbf{C})$. In order to examine the conjecture, we will study the pinching problem for the Gaussian curvature of the generalized Veronese surfaces in $\mathbf{P}^n(\mathbf{C})$, which is a complex-version of the Simon's conjecture discussed in Chapter I. Note that the formulas in Lemma 10.1 imply correct pinching value for the Gaussian curvature K. In fact, we obtain the following

Theorem 10.2. *Let* $\mathbf{X}$ *be a Kaehler manifold of complex dimension* n *of positive constant holomorphic sectional curvature* 4ρ *and* $\mathbf{M}$ *a complete connected oriented* 2*-dimensional Riemannian manifold. Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be a full isometric minimal immersion with constant Kaehler function* $\cos(\alpha)$*, which is neither holomorphic, anti-holomorphic nor totally real. Suppose that there exists a positive integer* m *such that each point of* $\mathbf{M}$ *is a J-regular point of order* $(m+1)$ *and that the Gaussian curvature* K *of* $\mathbf{M}$ *is strictly positive and satisfies*

$$\mathrm{K} \geq \frac{2\{1-(2m+3)\cos(\alpha)\}\rho}{(m+1)(m+2)}$$

on $\mathbf{M}$*. Then* K *is constant on* $\mathbf{M}$*. When* $\mathbf{X}$ *is* $\mathbf{P}^n(\mathbf{C})$*, such an immersion* x *is congruent to* $\varphi_{n,m+1}$*.*

Proof. By (10.11) and the assumptions, $(\mathcal{C}_m^2 \sum_\lambda |c_{\lambda,2[m-1]}|^2)$ is a non-zero subharmonic function on a compact manifold $\mathbf{M}$. Hence it is constant on $\mathbf{M}$. This shows

$$\mathrm{K} = \frac{2\{1-(2m+3)\cos(\alpha)\}\rho}{(m+1)(m+2)}.$$

By Ohnita's theorem [31] (cf. Section 7), we get Theorem 10.2. q.e.d.

Corollary 10.3. *Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be as in Theorem* 10.2*. If* $\mathbf{M}$ *is a J-regular manifold and the Gaussian curvature* K *satisfies*

$$\frac{2\{1-(2m+1)\cos(\alpha)\}\rho}{m(m+1)} > \mathrm{K} \geq \frac{2\{1-(2m+3)\cos(\alpha)\}\rho}{(m+1)(m+2)}$$

on $\mathbf{M}$*, then we have*

$$\mathrm{K} = \frac{2\{1-(2m+3)\cos(\alpha)\}\rho}{(m+1)(m+2)}.$$

Proof. By the J-regularity of $\mathbf{M}$ and the assumption, we have $\sum_\lambda |c_{\lambda,2[m-1]}|^2 \neq 0$ on $\mathbf{M}$. Hence each point of $\mathbf{M}$ is a J-regular point of order $(m+1)$. By Theorem 10.2, Corollary 10.3 then follows. q.e.d.

Remark. In Chapter II we only consider the case that the Gaussian curvature of a minimal surface in $\mathbf{P}^n(\mathbf{C})$ is strictly positive. If there exists a point p of $\mathbf{M}$ at which $\{1-(2m+3)\cos(\alpha)\} = 0$, then we have $\{1-(2m+3)\cos(\alpha)\} = 0$ everywhere on $\mathbf{M}$. By (10.11) we get K=0 , which contradicts to the assumption stated above. Hence it hold that $\{1-(2m+3)\cos(\alpha)\} > 0$.

Theorem 10.4. *Let* $\mathbf{X}$ *be a Kaehler manifold of complex dimension* n *of positive constant holomorphic sectional curvature* 4ρ *and* $\mathbf{M}$ *a complete connected oriented 2-dimensional Riemannian manifold such that the Gaussian curvature* K *is strictly positive. Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be a full isometric minimal immersion with constant Kaehler function* $\cos(\alpha)$*, which is neither holomorphic, anti-holomorphic nor totally real, where we put* $s = [n/2] - 1$ *(*$[a]$ *denotes the integer part of* a*). Suppose that* $\mathbf{M}$ *is a* J*-regular manifold and that* K *satisfies*

$$\mathrm{K} \geq \frac{2\{1-(2s+3)\cos(\alpha)\}\rho}{(s+1)(s+2)},$$

on $\mathbf{M}$*, then* K *is constant on* $\mathbf{M}$*. When* $\mathbf{X}$ *is* $\mathbf{P}^n(\mathbf{C})$*, we have that* x *is congruent to either* $\varphi_{n,1}, \ldots, \varphi_{n,s}$ *or* $\varphi_{n,s+1}$*.*

Proof. We inductively prove Theorem 10.4. The cases when each point of $\mathbf{M}$ is a J-regular point of order 1 and 2 have already been proved (cf. Theorem 7.1 and Theorem 7.4).

Let $p \in \mathbf{M}$ be a J-regular point of order $(m+1)(\geq 3)$. If there exists an open neighbourhood U of p such that K$< 2\{1-(2m+3)\cos(\alpha)\}\rho/(m+1)(m+2)$ on U, then we have $\sum |c_{\lambda,2[m]}|^2 \neq 0$ at some point $p' \in$U. In fact, assume that $\sum_\lambda |\mathcal{C}_m c_{\lambda,2[m]} + \bar{A}_m c_{\lambda,2[m-1]}|^2 = 0$ on U. This means that $d(\mathcal{C}_m^2 \sum_\lambda |c_{\lambda,2[m-1]}|^2) = 0$. So we get that $(\mathcal{C}_m^2 \sum_\lambda |c_{\lambda,2[m-1]}|^2)$ is non-zero constant on U by the regularity of $\mathbf{M}$. By (10.11) we have K $= 2\{1-(2m+3)\cos(\alpha)\}\rho/(m+1)(m+2)$, which contradicts the assumption on K. Hence $(\mathcal{C}_m^2 \sum_\lambda |c_{\lambda,2[m-1]}|^2)$ is a non-constant function. By

(10.6) we may assume that $c_{2m+1,2[m-1]} \neq 0$ at a point $p' \in$U . By the fifth formula of (10.8) we get that $\sum |c_{\lambda,2[m]}|^2 \neq 0$ at the $p' \in$U.

This shows that the $(m+2)$-th normal vectors $V_1^{(m+2)}$ and $V_2^{(m+2)}$ do not vanish at p' (cf.(9.11)) and p' is a J-regular point of order $k > (m+1)$. Among all possible values, $s = [n/2] - 1$ is the maximal one. Hence we can reduce a proof of the Theorem 10.4 to the case of $m = s$. We assume that each point of M is a J-regular point of order s. If $(\mathcal{C}_{s-1}^2 \sum_\lambda |c_{\lambda,2[s-2]}|^2)$ is non-constant on M, then each point of M is a J-regular point of order $(s+1)$. By Theorem 10.2 we see that K $= 2\{1-(2s+3)\cos(\alpha)\}\rho/(s+1)(s+2)$. If $(\mathcal{C}_{s-1}^2 \sum_\lambda |c_{\lambda,2[s-2]}|^2)$ is constant on M, then K $= 2\{1-(2s+1)\cos(\alpha)\}\rho/s(s+1)$. q.e.d.

Theorem 10.4 shows that the conjecture by Bolton et al. partially holds.

Chapter III
Surfaces with parallel mean curvature vector in $\mathbf{P}^2(\mathbf{C})$

The set of surfaces with parallel mean curvature vector field in a Riemannian manifold, which includes all minimal surfaces in the manifold, has been studied by many geometers. Especially, Chen [12] and Yau [39] studied them in the case that the ambient space is an n-dimensional real space form $\bar{\mathrm{M}}^n(c)$ of constant sectional curvature c. They proved that if $x : \mathrm{M} \longrightarrow \bar{\mathrm{M}}^n(c)$ is an isometric immersion with parallel mean curvature vector of a two dimensional Riemannian manifold M into $\bar{\mathrm{M}}^n(c)$, then $x(\mathrm{M})$ is one of the following surfaces: (1) *a minimal surface in* $\bar{\mathrm{M}}^n(c)$, (2) *a minimal surface of a small hypersphere of* $\bar{\mathrm{M}}^n(c)$, *and* (3) *a surface with constant mean curvature in a* 3*-sphere of* $\bar{\mathrm{M}}^n(c)$. This shows that the study of surfaces in $\bar{\mathrm{M}}^n(c)$ with parallel mean curvature vector is reduced to that of minimal surfaces except the case (3).

On the other hand, concerning the surfaces with parallel mean curvature vector in a complex space form, we know several minimal surfaces in the n-dimensional complex projective space $\mathbf{P}^n(\mathbf{C})$ with the Fubini-Study metric of constant holomorphic sectional curvature 4ρ. Moreover, many results characterizing them have been obtained (cf. [20],[24],[30],[34],[35]). However when we concern with non-minimal surfaces in $\mathbf{P}^n(\mathbf{C})$ with parallel mean curvature vector, not many examples are known so far, even for $n = 2$.

In Section 5 of Chapter II, we developed a local theory of surfaces in $\mathbf{P}^n(\mathbf{C})$ by using the Kaehler function. By applying it, in this Chapter we shall study non-minimal immersions $x : \mathrm{M} \longrightarrow \mathbf{P}^2(\mathbf{C})$ with parallel mean curvature vector by applying it. In fact, in Section 11 we obtain basic formulas for such surfaces in a 2-dimensional Kaehler manifold of constant holomorphic sectional curvature 4ρ. Then, in Sections 12 and 13, we prove local existence theorems of such immersions. Finally, in Section 14 we determine isometric immersions with parallel mean curvature vector field of a Riemannian 2-manifold with constant Gaussian curvature into $\mathbf{P}^2(\mathbf{C})$. Theorem 14.2 extends a theorem by Ludden, Okumura and Yano [30].

11. The fundamental theorem of surfaces in a Kaehler manifold

Let $\mathbf{X}$ be a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature 4ρ. Let $\{\omega_\alpha\}$ be a local field of unitary coframes on $\mathbf{X}$, so that the Kaehler metric is represented by $\sum \omega_\alpha \overline{\omega}_\alpha$. Here and in what follows, we will agree on the following range of indices: $1 \leq \alpha, \beta, \gamma \leq 2$. We denote by $\omega_{\alpha\beta}$ the unitary connection forms with respect to $\{\omega_\alpha\}$. The structure equations of $\mathbf{X}$ are given by (5.1).

Let ($\mathbf{M}$, ds^2) be an oriented connected 2-dimensional Riemannian manifold. The tangent bundles of $\mathbf{M}$ and $\mathbf{X}$ are denoted by $\mathrm{T}\mathbf{M}$ and $\mathrm{T}\mathbf{X}$, respectively. Let $x : \mathbf{M} \longrightarrow \mathbf{X}$ be an isometric immersion of $\mathbf{M}$ into $\mathbf{X}$. By means of the differential $\mathrm{d}x$ we may consider $\mathrm{T}\mathbf{M}$ as a subbundle of the induced bundle $x^*\mathrm{T}\mathbf{X}$ over $\mathbf{M}$, so that we get the orthogonal decomposition $x^*\mathrm{T}\mathbf{X} = \mathrm{T}\mathbf{M} \oplus \mathrm{N}\mathbf{M}$, where $\mathrm{N}\mathbf{M}$ denotes the normal bundle of x.

Let $\{\mathrm{e}_1, \mathrm{e}_2\}$ be an oriented orthonormal local frame on $\mathbf{M}$. Let $< , >$ denote the Riemannian metric of $\mathbf{X}$ induced by the Kaehler metric and J the complex structure of $\mathbf{X}$. The *Kaehler function* $\cos(\alpha)$ on $\mathbf{M}$ is defined by

$$\cos(\alpha) = < Je_1, e_2 >,$$

which is independent of the choice of oriented orthonormal frames on $\mathbf{M}$. The immersion is said to be holomorphic if $\cos(\alpha) = 1$ on $\mathbf{M}$, anti-holomorphic if $\cos(\alpha) = -1$ on $\mathbf{M}$, and totally real if $\cos(\alpha) = 0$ on $\mathbf{M}$.

Recall that in Section 5 it was assumed that x is neither holomorphic nor anti-holomorphic at any point of $\mathbf{M}$. In this section, we also assume the same conditions on x, and use the formulas obtained in Section 5. Let $\mathbf{H}$ be the mean curvature vector field of x, which is defined by

$$\mathbf{H} = \frac{1}{2}\sum_{\alpha,i} h_{\alpha ii} e_\alpha,$$

where $h_{\alpha ij}$'s are the components of the second fundamental form of x (cf. Section 1), and e_i and e_α are the adapted frames along x. x is called minimal if $\mathbf{H} = 0$ on $\mathbf{M}$. Let $\mathbf{D}^\perp$ denote the connection of the normal bundle NM. If

$$\mathbf{D}^\perp \mathbf{H} = 0$$

on $\mathbf{M}$, then $\mathbf{H}$ is called the *parallel* mean curvature vector field.

We assume that $\mathbf{H} \neq 0$, $\mathbf{D}^\perp\mathbf{H} = 0$ on $\mathbf{M}$, and the Kaehler function is $\cos(\alpha)$. We can construct a unique system of global orthonormal vector fields $\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3,\tilde{e}_4\}$ along $\mathbf{M}$ such that $\tilde{e}_1$ and $\tilde{e}_2$ are tangent to $\mathbf{M}$ in the following way: First we put $\tilde{e}_3 = -\mathbf{H}/ \parallel \mathbf{H} \parallel$, then the normal vector field $\tilde{e}_4$ of NM is uniquely determined by choosing it to be compatible with the fixed orientations of $\mathbf{M}$ and $\mathbf{X}$. The system of vectors $\{\tilde{e}_3,\ \tilde{e}_4,\ J\tilde{e}_3,\ J\tilde{e}_4\}$ is linearly independent, because x is neither holomorphic nor anti-holomorphic. We have the identity

$$\cos(\alpha) =< J\tilde{e}_4, \tilde{e}_3 >$$

which is easily proved by using the fact that $\cos(\alpha)$ is independent of the choice of the oriented orthonormal frame on $\mathbf{M}$. By using the Schmidt orthonormalization, we get a new frame $\{\tilde{e}_1,\ \tilde{e}_2\}$ on $\mathbf{M}$ defined by

$$\tilde{e}_1 = \cot(\alpha)\tilde{e}_3 - \operatorname{cosec}(\alpha)J\tilde{e}_4,$$

$$\tilde{e}_2 = \operatorname{cosec}(\alpha)J\tilde{e}_3 + \cot(\alpha)\tilde{e}_4.$$

It is easy to see that $\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3,\tilde{e}_4\}$ is an adapted frame on $\mathbf{M}$ in $\mathbf{X}$, that is, $\tilde{e}_1$ and $\tilde{e}_2$ are sections on TM and $\tilde{e}_3$ and $\tilde{e}_4$ are sections on NM. Moreover, we define vector fields e_1 and e_3 as follows:

$$e_1 = \frac{1}{2}\sec(\frac{\alpha}{2})(\tilde{e}_1 - J\tilde{e}_2),$$

$$e_3 = \frac{1}{2}\mathrm{cosec}(\frac{\alpha}{2})(\tilde{e}_1 + J\tilde{e}_2),$$

Put

$$e_2 = Je_1 \quad \text{and} \quad e_4 = Je_3.$$

Then $\{e_1,e_2,e_3,e_4\}$ is a J-canonical frame along x (cf. Section 5). We extend $\{\tilde{e}_A\}$ and $\{e_A\}$ to a neighbourhood of **M** in **X**, where A,B and C run from 1 through 4.

Let $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$ be the dual coframes of $\{\tilde{e}_A\}$ and $\{e_A\}$ respectively. Let $\tilde{\theta}_{AB}$ and θ_{AB} be the Riemannian connection forms with respect to the canonical 1-forms $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$, respectively and put

$$\omega_\alpha = \theta_{2\alpha-1} + i\theta_{2\alpha},$$

$$\omega_{\alpha\beta} = \theta_{2\alpha-1,2\beta-1} + i\theta_{2\alpha,2\beta-1}, \quad \text{where} \quad i = \sqrt{-1}.$$

Then we have the following relations (cf. (5.5)):

$$\begin{aligned}
\tilde{\theta}_{12} &= i(\cos^2(\frac{\alpha}{2})\omega_{11} - \sin^2(\frac{\alpha}{2})\omega_{22}),\\
\tilde{\theta}_{34} &= -i(\sin^2(\frac{\alpha}{2})\omega_{11} - \cos^2(\frac{\alpha}{2})\omega_{22}),\\
\tilde{\theta}_{13}+i\tilde{\theta}_{23} &= -\,\omega_{12} - \frac{1}{2}(d\alpha - \sin(\alpha)(\omega_{11} + \omega_{22})),\\
\tilde{\theta}_{14}+i\tilde{\theta}_{24} &= i\{\omega_{12} - \frac{1}{2}(d\alpha - \sin(\alpha)(\omega_{11} + \omega_{22}))\}.
\end{aligned} \tag{11.1}$$

We denote the restriction of $\{\tilde{\theta}_A\}$ to **M** by the same letters and put

$$\phi = \tilde{\theta}_1 + i\tilde{\theta}_2.$$

By the assumptions, $\tilde{e}_3$ is a parallel vector field along **M**, hence so is $\tilde{e}_4$. This implies

$$\tilde{\theta}_{34} = 0. \tag{11.2}$$

It is then proved that there exists a positive number b, complex-valued smooth functions a and c defined locally on **M**, which satisfy the followings (cf. (6.1), (6.2), (6.3), (6.4) and (6.5)):

$$\tilde{\theta}_{12} = i\cot(\alpha)\{(a-b)\phi - (\bar{a}-b)\bar{\phi}\},$$

$$d\alpha = (a+b)\phi + (\bar{a}+b)\bar{\phi},$$

$$(11.3) \qquad (da - ia\tilde{\theta}_{12})\wedge\phi = -\left\{\cot(\alpha)(\bar{a}-b)a + \frac{3}{4}\rho\sin(2\alpha)\right\}\phi\wedge\bar{\phi},$$

$$(dc + 3ic\tilde{\theta}_{12})\wedge\bar{\phi} = \cot(\alpha)(b - a)c\phi\wedge\bar{\phi},$$

$$\mathbf{H} = -2b\tilde{e}_3.$$

The third and fourth formulas of (11.3) are the Codazzi equations of x.

Denoting by K the Gaussian curvature of M, the Gauss equation of x is written as

$$(11.4) \qquad \mathrm{K} = 6\rho\cos^2(\alpha) - 4(|\,a\,|^2 - b^2).$$

Let K_N be the normal curvature of x defined by

$$d\tilde{\theta}_{34} = -\mathrm{K}_N\tilde{\theta}_1\wedge\tilde{\theta}_2.$$

By taking the exterior derivative of the second formula of (11.1) and using the formula (6.1) in Chapter II, we have

$$\mathrm{K}_N = (3\cos^2(\alpha) - 1)\rho + 2(|\,c\,|^2 - |\,a\,|^2).$$

Since the normal curvature of x vanishes, we get

$$(11.5) \qquad |\,c\,|^2 = |\,a\,|^2 - \frac{\rho}{2}(3\cos^2(\alpha) - 1)).$$

Combining formulas (11.4) and (11.5), we get

$$(11.6) \qquad \mathrm{K} = (1 + 3\cos^2(\alpha))\rho - 2(|\,a\,|^2 - 2b^2 + |\,c\,|^2).$$

For a neighbourhood U of a point of M, there exists an isothermal coordinate

$$z = u + iv \quad \text{such that} \quad ds^2 = \lambda^2|dz|^2,$$

where λ is a positive function defined on U, and we have

$$\phi = \lambda dz.$$

Then the set of the first three formulas of (11.3) is rewritten as the following system of differential equations:

$$
\begin{aligned}
&\frac{\partial\lambda}{\partial z} = -\lambda^2\cot(\alpha)(a-b),\\
(11.7)\qquad &\frac{\partial\alpha}{\partial z} = \lambda(a+b),\\
&\frac{\partial a}{\partial \bar{z}} = \lambda\{2\cot(\alpha)(\bar{a}-b)a + \frac{3}{4}\rho\sin(2\alpha)\}.
\end{aligned}
$$

By using (11.7), we have that

$$
(11.8)\qquad \frac{\partial^2\lambda}{\partial z\partial\bar{z}} = \frac{\partial^2\lambda}{\partial\bar{z}\partial z} \quad \text{if and only if} \quad \bar{a} = a.
$$

Therefore a is a real-valued function defined locally on $\mathbf{M}$. This implies that λ, α and a are functions of single variable, and (11.7) is seen to be a system of ordinary differential equations. Consequently, if $\mathbf{M}$ is a non-minimal surface with parallel mean curvature vector field in X, then there exists a positive number b and real-valued smooth functions of single variable λ, a and c which are defined locally on $\mathbf{M}$ and satisfy the system of ordinary differential equations (cf. 12.1).

Remark. The fourth formula of (11.3) is equivalent to the equation

$$
(11.9)\qquad \frac{\partial(\lambda^2 c)}{\partial z} = 0.
$$

In the next section, we shall consider a converse problem to the result obtained above, that is, local existence problem for non-minimal surfaces in X with parallel mean curvature vector field. To this end, we need the fundamental theorem of surfaces theory in X. When X is a real space form, the fundamental theorem of submanifolds is well-known (cf. M. Dajczer, *Submanifolds and immersions*, Publish or Perish, (1990)). On the other hand, for a surface in a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature 4ρ, the following fundamental theorem is proved by Eschenburg et al.[20], which is essential use in this Chapter:

Theorem 11.1 ([20]). *Let* $(\mathbf{M}, ds^2)$ *be a connected, simply connected 2-dimensional Riemannian manifold. Given complex-valued 1-forms* $\omega_1, \omega_2, \omega_{11}, \omega_{22}$ *and* ω_{12} *defined on* $\mathbf{M}$ *satisfying the structure equations* (5.1) *and*

$$ds^2 = \omega_1\bar{\omega_1} + \omega_2\bar{\omega_2},$$

there exist an isometric immersion $x : \mathbf{M} \longrightarrow \mathbf{X}$ *and a unitary frame* $\{E_1, E_2\}$ *along* x *such that* $\{\omega_1, \omega_2\}$ *is the unitary coframe of* $\{E_1, E_2\}$, *and* ω_{11}, ω_{22} *and* ω_{12} *are the unitary connection forms with respect to* $\{\omega_1, \omega_2\}$.

12. Local existence of surfaces in $\mathbf{P}^2(\mathbf{C})$

It was B.Y.Chen who constructed surfaces with constant mean curvature in a 3-dimensional real space form (cf. [12], p.121). In Theorem 3.11 of [20] Eschenburg et al. proved a local existence theorem for minimal surfaces in $\mathbf{P}^2(\mathbf{C})$. In this section, we consider a corresponding local existence theorem of a non-minimal surface with parallel mean curvature vector field in a complex 2-dimensional Kaehler manifold. The following theorem shows a method of the local construction of such surfaces.

Theorem 12.1. *Let* b *and* ρ *be real numbers* ($b > 0$), *and* λ, α *and* a *be real-valued smooth functions of single variable* u *defined on an interval* I, *which satisfy the following system of ordinary differential equations:*

$$\begin{aligned} \frac{d\lambda}{du} &= -\lambda^2 \cot(\alpha)(a-b), \\ \frac{d\alpha}{du} &= \lambda(a+b), \qquad (12.1) \\ \frac{da}{du} &= \lambda\{2\cot(\alpha)(a-b)a + \frac{3}{4}\rho\sin(2\alpha)\}. \end{aligned}$$

Let $\mathbf{M}$ *be an open domain of* (u,v)*-plane contained in* $I \times (-1, 1)$. *Define*

$$ds^2 = \lambda^2(du^2 + dv^2)$$

on $\mathbf{M}$. *Then there exists an isometric immersion* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *of* $\mathbf{M}$ *into a complex 2-dimensional Kaehler manifold* $\mathbf{X}$ *of constant holomorphic sectional curvature* 4ρ *which satisfies the following:*

(1) *x has a non-zero parallel mean curvature vector field whose length is $2b$,*

(2) *the Kaehler function of x is $\cos(\alpha)$,*

(3) *the second fundamental form of x is explicitly written in terms of a, b, λ and α.*

Proof. Let (r,s,t) be the standard coordinate of $\mathbf{R}^3$ and $\mathbf{D}$ a domain in $\mathbf{R}^3$ such that $r > 0$ and $0 < s < \pi$. We define a $\mathbf{R}^3$-valued function $f(r,s,t)$ on $\mathbf{D}$ by

$$f(r,s,t) = \begin{pmatrix} -r^2 \cot(s)(t-b) \\ r(t+b) \\ r\{2\cot(s)(t-b)t + 3\rho \sin(2s)/4\} \end{pmatrix}.$$

$f(r,s,t)$ has continuous partial derivatives on $\mathbf{D}$, so that it satisfies Lipschitz condition on $\mathbf{D}$. Hence, a solution of the system (12.1) exists and is unique under preassigned initial conditions.

Let (λ, α, a) be a solution of (12.1) and we put

$$z = u + iv \text{ and } \phi = \lambda dz.$$

The fourth formula in (11.3) gives

$$(12.2) \qquad 2i|c|^2 d\tau = \{4|c|^2 \cot(\alpha)(a-b) - A_1\}\phi - \{4|c|^2 \cot(\alpha)(a-b) - \bar{A}_1\}\bar{\phi},$$

where we put $d|c|^2 = A_1\phi + \bar{A}_1\bar{\phi}$. By using (11.5) and (12.2) we can define a function c. Then it is proved that $\lambda^2 c$ is anti-holomorphic which is equivalent to (11.9). We define $\omega_1, \omega_2, \omega_{11}, \omega_{22}$ and ω_{12} on $\mathbf{D}$ as follows:

$$\omega_1 = \cos(\frac{\alpha}{2})\phi,$$

$$
\begin{aligned}
&\omega_2 = \sin(\frac{\alpha}{2})\bar{\phi}, \\
(12.3) \qquad &\omega_{11} = \frac{1}{2}\cot(\frac{\alpha}{2})\{(a-b)\phi - (a-b)\bar{\phi}\}, \\
&\omega_{22} = \frac{1}{2}\tan(\frac{\alpha}{2})\{(a-b)\phi - (a-b)\bar{\phi}\}, \\
&\omega_{12} = -\bar{\omega}_{21} = b\phi + c\bar{\phi}.
\end{aligned}
$$

Note that these satisfy (5.1) because of (12.1). Therefore, by Theorem 11.1, there exists an isometric immersion $x : \mathbf{M} \longrightarrow \mathbf{X}$ which has a non-zero, parallel mean curvature vector field and $\cos(\alpha)$ as the Kaehler function. The second fundamental form of x is explicitly written in terms of a,b,λ and α by (6.2) of Chapter II. q.e.d.

13. Associated family of isometric immersions

It is well-known that there exists a one-parameter family of isometric surfaces in $\mathbf{R}^3$ with the same constant mean curvature. The following theorem shows that an analogous property holds in the case that the ambiant space is a 2-dimensional complex space form $\mathbf{X}$ and that the mean curvature vector field $\mathbf{H}$ of an immersed surface is parallel. Note that Eschenburg et al. [20] have proved that there exists a one-parameter family of isometric minimal immersions of a simply connected surface into $\mathbf{P}^2(\mathbf{C})$ with the same normal curvature and Kaehler function. Theorem 13.1 is an extension of Theorem B in [20] stated above.

Theorem 13.1. *Let* $\mathbf{X}$ *be a complex 2-dimensional Kaehler manifold of constant holomorphic sectional curvature* 4ρ *and* $(\mathbf{M}, ds^2)$ *be a simply connected oriented 2-dimensional Riemannian manifold. Let* $x : \mathbf{M} \longrightarrow \mathbf{X}$ *be an isometric immersion of* $\mathbf{M}$ *into* $\mathbf{X}$ *with non-zero parallel mean curvature vector field* $\mathbf{H}$ *and* $\cos(\alpha)$ *the Kaehler function. Assume that the immersion* x *is neither holomorphic nor anti-holomorphic. Then there exists a one-parameter family of isometric immersions* $x_t : \mathbf{M} \longrightarrow \mathbf{X}$, $t \in (-\pi, \pi)$, *which satisfies the following properties:*

(1) $x_0 = x$,

(2) x_t *is isomertic to* x *for each* t,

(3) $\|\mathbf{H}_t\| = \|\mathbf{H}\| \neq 0$, *where* $\mathbf{H}_t$ *denotes the mean curvature vector field of* x_t,

(4) $\mathbf{D}_t^{\perp}\mathbf{H}_t = 0$, *where* $\mathbf{D}_t^{\perp}$ *is the normal connection of* x_t,

(5) $\cos(\alpha_t) = \cos(\alpha)$,

(6) x_t *is not congruent to each other.*

Proof. By the assumptions, we can use the results in Section 11. The first formula in (11.3) implies

$$a\phi \wedge \bar{\phi} = i\tan(\alpha)\tilde{\theta}_{12} \wedge \bar{\phi} + b\phi \wedge \bar{\phi}.$$

This shows that the real valued function a is uniquely determined by the Riemannian metric ds^2, the mean curvature vector field $\mathbf{H}$ and $\cos(\alpha)$. By (11.5), $|c|^2$ is also uniquely determined by ds^2, $\mathbf{H}$ and $\cos(\alpha)$. We put

$$c = |c|e^{i\tau}, \quad 0 \leq \tau < 2\pi,$$

where τ is a real-valued function on $\mathbf{M}$. The formula (12.2) shows that τ is uniquely determined by ds^2, $\mathbf{H}$ and $\cos(\alpha)$, up to additive constants. Hence, if we put

$$c_t = ce^{it} \text{ for some } t\epsilon\ (-\pi, \pi),$$

then c_t also satisfies the fourth formula in (11.3). We put

$$\omega_{12} = -\bar{\omega}_2 = b\varphi + c_t\bar{\varphi},$$

and the other connection forms are defined similarly as in (12.3). Then $\omega_1, \omega_2, \omega_{11}, \omega_{22}$ and ω_{12} satisfy (5.1) for each t. By Theorem 11.1, there exists an isometric immersion $x_t : \mathbf{M} \longrightarrow \mathbf{X}$ for which the adapted frame

$$\{\tilde{e}_1(t), \tilde{e}_2(t), \tilde{e}_3(t), \tilde{e}_4(t)\}$$

along x_t satisfies

$$\mathbf{H}_t = -2b\tilde{e}_3(t), \quad \mathbf{D}_t^{\perp}\mathbf{H}_t = 0,$$

and $\cos(\alpha)$ is the Kaehler function of x_t for each t. q.e.d.

Corollary 13.2. *Let $x_i : \mathbf{M} \longrightarrow \mathbf{X}$ $(i = 1, 2)$ be an isometric immersion with non-zero, parallel mean curvature vector field $\mathbf{H}_i$ and the Kaehler function $\cos(\alpha_i)$. Assume that x_i are neither holomorphic nor anti-holomorphic and that x_1 is isometric to x_2. Then x_1 is congruent to x_2 if and only if*

$$\cos(\alpha_1) = \cos(\alpha_2), \quad \| \mathbf{H}_1 \| = \| \mathbf{H}_2 \| \quad \text{and} \quad c_1 = c_2.$$

14. Complete flat surfaces with parallel mean curvature vector

In this section we apply the results obtained in Chapter III to the case that $(\mathbf{M}, ds^2)$ is a Riemannian manifold of constant Gaussian curvature. As a result, we determine all isometric immersions of the $(\mathbf{M}, ds^2)$ into $\mathbf{P}^2(\mathbf{C})$ with parallel mean curvature vector field. We put ρ=1 for simplicity.

Let $\mathbf{M}^2[\mathrm{K}]$ denote an oriented connected 2-dimensional Riemannian manifold of constant Gaussian curvature K and $x : \mathbf{M}^2[\mathrm{K}] \longrightarrow \mathbf{P}^2(\mathbf{C})$ be an isometric immersion whose mean curvature vector field $\mathbf{H}$ is parallel but non-vanishing. Differentiating (11.4) and using $\bar{a} = a$, we have

$$2a\frac{da}{du} + 3\cos(\alpha)\sin(\alpha)\frac{d\alpha}{du} = 0. \tag{14.1}$$

Since the system (12.1) is valid for the immersion x, the formulas (14.1) and (12.1) give

$$\cos(\alpha) \equiv 0$$

or

$$3\sin^2(\alpha) = -\frac{4a^2(a-b)}{2a+b}.$$

It follows from these formulas and the Gauss equation (11.4) that a is constant, $\alpha = \pi/2$ and hence $\mathrm{K} = 0$. In consequence, we obtain the following.

Proposition 14.1. *Let* $\mathbf{M}^2[\mathrm{K}]$ *be an oriented 2-dimensional Riemannian manifold of constant Gaussian curvature* K *and* $x : \mathbf{M}^2[\mathrm{K}] \longrightarrow \mathbf{P}^2(\mathbf{C})$ *an isometric immersion such that the mean curvature vector field is parallel and not zero. Then* x *is totally real and* $\mathrm{K} = 0$.

Now we are going to determine isometric immersions with parallel mean curvature vector field of a complete flat surface into $\mathbf{P}^2(\mathbf{C})$. Let $\mathbf{R}^2$ be the Euclidean 2-plane with the standard flat metric $ds^2 = du^2 + dv^2$. We put

$$\phi = dz \text{ and } z = u + iv.$$

Let $x : \mathbf{R}^2 \longrightarrow \mathbf{P}^2(\mathbf{C})$ be an isometric immersion with non-zero parallel mean curvature vector field. It follows from Proposition 14.1 that the immersion x must be totally real and $\alpha = \pi/2$. By (11.3), we have $a = -b$. By (11.5) and (12.2), c is a complex constant with

$$| c |^2 = b^2 + \frac{1}{2}.$$

On account of Theorem 13.1, we may assume that c is real. Therefore we have

$$\begin{aligned}
\omega_1 &= \frac{1}{\sqrt{2}}\phi, \\
\omega_2 &= \frac{1}{\sqrt{2}}\bar{\phi}, \\
\omega_{11} &= -\, b(\phi - \bar{\phi}), \\
\omega_{22} &= -\, b(\phi - \bar{\phi}), \\
\omega_{12} &= -\, \bar{\omega}_{21} = b\phi + c\bar{\phi}.
\end{aligned} \tag{14.2}$$

where b and c are real constants such that $b > 0$ and $c = \sqrt{b^2 + 1/2}$.

We can solve the system (14.2) in the same way as in Kenmotsu [24, p.p.679–681]: Let $\lambda_i, i = 0, 1, 2$, be the eigenvalues of the matrix A defined by

$$A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & b & c \\ -\frac{1}{\sqrt{2}} & -b & b \end{pmatrix}. \tag{14.3}$$

It is easy to see that, if necessary renumbering λ_i, λ_0 is a non-zero real number which is not rational, λ_1 is a complex number which is not real and λ_2 is the complex conjugate of λ_1. Put

$$\mathrm{G} = \{(exp(\lambda_i z - \bar{\lambda}_i \bar{z})\delta_{ij}) \mid z = u + iv, (u, v) \epsilon \mathbf{R}^2\}.$$

Then x $(\mathbf{R}^2)$ is an orbit of the abelian Lie subgroup G of the unitary group U(3). We remark that G is homeomorphic to the cylinder $\mathrm{S}^1 \times \mathbf{R}^1$.

Summarizing our results of this section, we obtain the following Theorem.

Theorem 14.2. *Let $x : \mathbf{R}^2 \longrightarrow \mathbf{P}^2(\mathbf{C})$ be an isometric immersion with non-zero parallel mean curvature vector field. Then $x(\mathbf{R}^2)$ is an orbit of the abelian Lie subgroup* G *of* U(3) *and* G *is algebraically determined by the constant b, where $2b$ is the length of the mean curvature vector field* **H** *of x.*

It should be remarked that when x is minimal and totally real, this theorem was proved in [30].

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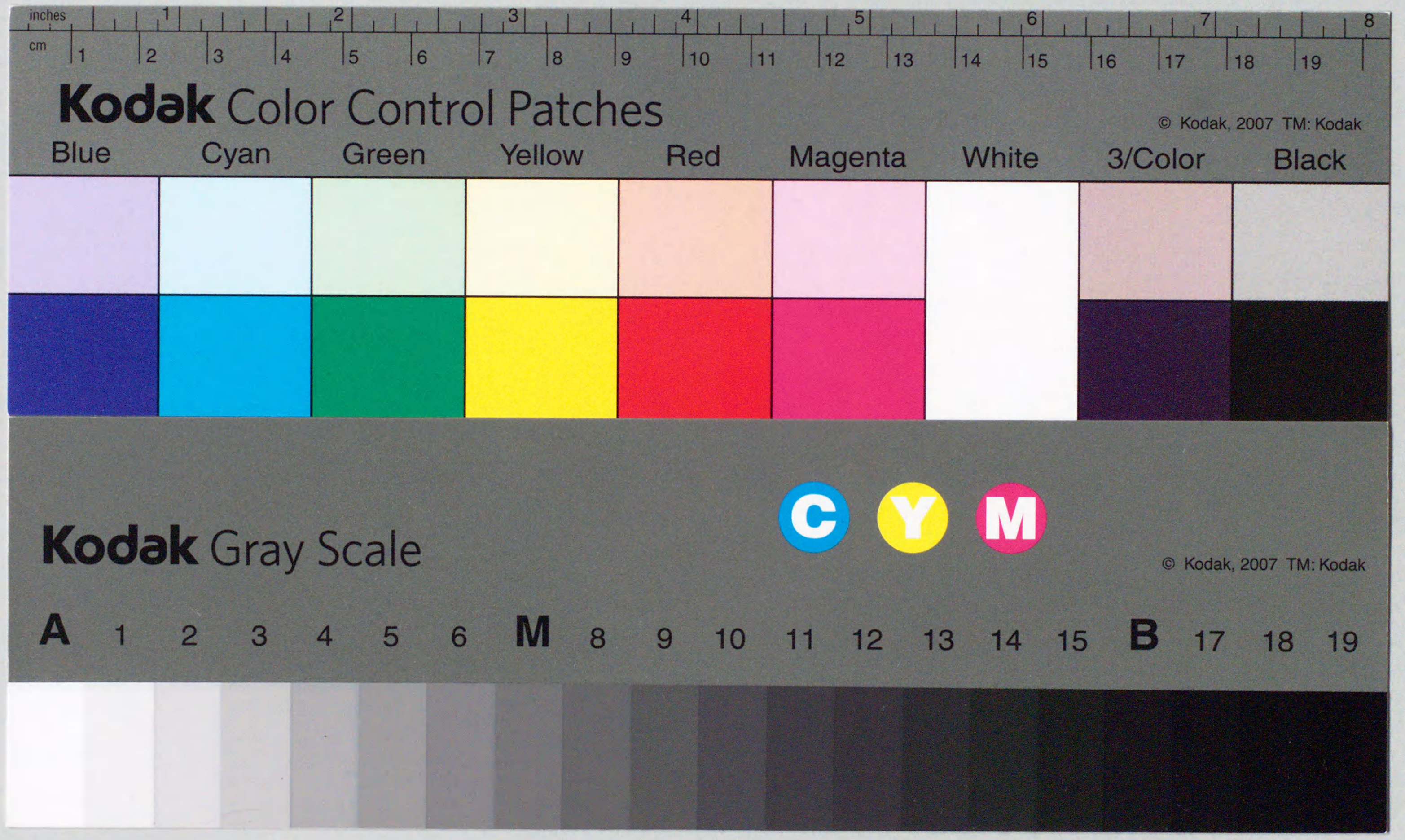

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